Ideas for final projects
Math 145, Spring 2001

You should expect your final paper to be around 10 pages long, depending on your topic, your font size and spacing, etc. Topics in $\boxed{\textit{boxes}}$ are particularly fundamental, as well as potentially useful to other projects. It might be nice to have seniors cover those topics, since it might be useful to have the entire class discuss them relatively early. Note, however, that you are certainly not limited to these topics.

Incidentally, if you are (eventually) looking for a senior thesis in math, this list is not a bad place to start.

First due date: You must choose a topic by **April 16**. Please choose as early as possible.

Aspects of hyperbolic geometry.

1. *Other models of* $\mathbf{H}^2$. There is a unit disk model (Stahl Ch. 13), an upper hemisphere model, and a "straight-line" model (Stahl Ch. 14). There is also a linear algebra approach using "orthogonal geometry," which can actually be used to see all of the models at once.

2. $\boxed{\mathbf{H}^3.}$ See Stahl Ch. 13 for an introduction. Note that there is a connection with $\mathbf{GL}_2(\mathbf{C})$ that does not exist in higher dimensions.

3. $\mathbf{H}^n$. This can be approached either via orthogonal geometry or circle inversions (conformal geometry).

Other geometries.

1. *Spherical geometry.* See Stahl Ch. 11 for an introduction in the case of $\mathbf{S}^2$. One can also consider $\mathbf{S}^3$, which is actually itself a group (!), or $\mathbf{S}^n$.

2. *n-dimensional Euclidean space.* There is a nice algebraic characterization of $\mathrm{Isom}(\mathbf{E}^n)$.

3. $\boxed{\textit{Orthogonal geometry.}}$ This is a general method of approach geometry, through the multilinear algebra of symmetric bilinear forms, and can be considered in a generality encompassing spherical, Euclidean, and hyperbolic geometry in n dimensions.

4. *Symplectic geometry.* This is like orthogonal geometry, except with skew-symmetric forms. There are interesting applications to physics.

5. *The Siegel upper half-plane.* This is a natural way of constructing a space whose isometry group is the symplectic group, and is a generalization of the Poincareé upper half-plane.

6. *Unitary geometry.* This is like orthogonal geometry, except over the complex numbers.

7. *Complex hyperbolic spaces.* This is a particularly interesting type of unitary geometry, the first case of which is actually the disk model of the hyperbolic plane.

8. *Projective geometry.* This is not geometry in the (metric) sense we have been discussing; this is more like the geometry of lines or spheres at infinity (e.g., $\hat{\mathbf{R}}$ and $\hat{\mathbf{C}}$). The topic has a very nice axiomatic development and some interesting and surprising theorems.

9. *The 3-dimensional geometries.* There are eight fundamental geometries in 3 dimensions, all of which are related to the 2-dimensional geometries we've studied in some way.

10. *Taxicab geometry.* This is the geometry of getting from place to place in, say, midtown Manhattan, or any other gridlike city. It can also be seen as the geometry of the group $\mathbf{Z} \times \mathbf{Z}$.

Geometry and topology.

1. *Riemannian geometry.* This is the study of (non-constant) curvature and generalizations of our definition of distance in the hyperbolic plane. See Stahl Ch. 12 for an overview; there are much better sources if you're interested.

2. *Cell complexes.* This is a generalization of the way in which our class makes surfaces out of polygons.

3. *The fundamental group.* One key to the connection between geometry and topology is, roughly speaking, the group of all loops in a space.

4. *Covering spaces.* This is sort of a "dual" approach to the ideas in the fundamental group; roughly speaking, the goal is to look at any space as a "quotient" of a space as much like $\mathbf{R}^n$ as possible.

5. *Riemann surfaces.* This topic looks at surfaces from the viewpoint of complex analysis.

6. *Hyperbolic knots.* In some sense, most knots can be analyzed by cutting them into pieces that are (sort of) quotients of hyperbolic space.

7. *Uniformization of surfaces.* Continuing the covering space idea, this theorem says that any surface is a "quotient" of either $\mathbf{S}^2$, $\mathbf{E}^2$, or $\mathbf{H}^2$. This can also be seen in terms of Riemann surfaces.

8. *Thurston's geometrization conjecture.* This is the 3-dimensional version of uniformization.

9. *Teichmüller spaces.* As a refinement of uniformization, one can study the family of all ways in which a surface can be uniformized.

10. *Manifolds.* One can look at (topological, differential, or complex) n-manifolds, which are to dimension n as surfaces are to dimension 2.

11. *Orbifolds.* A torus can be uniformized as a quotient of $\mathbf{E}^2$ by a group of translations. Orbifolds are, for example, spaces uniformized as a quotient of $\mathbf{E}^2$ by an arbitrary discrete group of isometries.

Geometric group theory.

1. *Discrete groups.* There is a certain amount of fundamental material to cover on the topic of groups acting discontinously on a geometry. Topics include: definitions, fundamental domains, examples.

2. *Quotients and tilings.* Covering spaces and orbifolds (see above) can also be seen in terms of symmetry groups of tilings.

3. *Coxeter groups.* These are discrete groups generated by reflections, and can be studied in the spherical, euclidean, and hyperbolic cases.

4. *Fuchsian groups.* Discrete subgroups of $\mathrm{Isom}(\mathbf{H}^2)$.

5. *Kleinian groups.* Discrete subgroups of $\mathrm{Isom}(\mathbf{H}^3)$. Somewhat more complicated than Fuchsian groups, and qualitatively different as well.

6. *Limit sets of Kleinian groups.* It is interesting to look at what a Kleinian group does on the boundary of $\mathbf{H}^3$; for example, fractal geometry plays an important role.

7. *The plane crystallographic/wallpaper groups.* Up to isomorphism, there are 17 types of discrete subgroups of $\mathrm{Isom}(\mathbf{E}^2)$ with compact quotient ("17 ways to make wallpaper").

8. *Bieberbach groups.* Discrete subgroups of $\mathrm{Isom}(\mathbf{E}^n)$. One might start with $n = 3$.

9. *δ-hyperbolic spaces.* Metric spaces that satisfy the thin triangles theorem turn out to be quite a bit like the hyperbolic plane.

10. *The Cayley graph of a group.* One may think of a group G as a geometry, and study the group G acting on the geometry G.

11. *Word-hyperbolic groups.* In particular, groups that are δ-hyperbolic as geometries turn out to be very interesting.

12. *CAT(0).* This property is an interesting "thin triangles theorem" generalizing Stahl p. 129, 24.

13. *Presentations and topology.* A group described by generators and relations has a topological space associated with it that can often be useful.

14. *The Poincaré polygon theorem.* This theorem gives a complete algebraic description of a discrete group given certain information about its fundamental domain. (Sort of a geometric version of the previous topic.)

15. *Trees and free groups.* Theorem: A group is free if and only if it acts freely (i.e., without fixed points) on a tree.

16. *Trees and free products.* Removing the "freely" part of the previous theorem gives a characterization of free products of groups.

17. *Schottky groups.* This is an way of getting free groups from a treelike construction in $\hat{\mathbf{C}}$.

Other.

1. *Modular functions and forms.* These functions on the upper half-plane with hyperbolic symmetries play an important role in the proof of Fermat's last theorem.

2. *General relativity.* Einstein's general theory of relativity is naturally set in one model of $\mathbf{H}^3$.

3. *The Smith chart.* $\mathbf{H}^2$ can be used to model characteristics of certain "black box" electrical circuits.

4. *Lie groups.* Lie groups are both groups and manifolds, and underlie much of what we have done in this class.

5. *Regular polyhedra.* There is a group-theoretical characterization of the Platonic solids (tetrahedron, cube, etc.).

6. *Regular polytopes.* These are the n-dimensional versions of the regular polyhedra, and can also be classified group-theoretically.

7. *Escher's regular division drawings.* Some of Escher's drawings are based on discrete groups of Euclidean, spherical, or hyperbolic isometries.

8. *Farey fractions.* This involves a tiling of the hyperbolic plane closely related to the action of $\mathbf{PSL}_2(\mathbf{Z})$ on rational numbers.

9. *Continued fractions.* This can be seen as an application of the action of $\mathbf{PSL}_2(\mathbf{Z})$ on rational numbers towards understanding "infinite fraction expansions" like

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$