Math 42 Final, Fall 2022

PRINT your name:

[Your exam can receive a 0 without a signature for this honor code] On my honor as a human being, I have not given / received or will give / receive any help on this exam that the instructor would not approve of. I understand that I will receive an F for the course if I break this promise.

SIGN your name:

Instructions:

1. You have 120 minutes to complete the exam.
2. For problems that say "explain" or "prove," you must show all of your work to receive full credit.
3. There are 10 problems for 10 points each (so 100 points total). You should have 11 pages.
4. No outside aid (cell phones, notes, books, calculators, use of internet and computer, or anything else not explicitly given permission for) are allowed at any time.
5. Good luck!

1 (10 pts.)

1. (5 pts) Find the negation of the English statement "Everyone in your class likes either cartoons, serious movies, or both."

2. (5 pts) Find the negation of the mathematical statement "$p \to (q \wedge (r \vee s))$."

2 (10 pts.)

1. (5 pts) Let $f : \{1, 2, 3\} \to \{red, blue\}$ be defined as $f(1) = red$, $f(2) = blue$, $f(3) = blue$. Is f injective? Is f surjective? Is f bijective? Explain all of your answers.

2. (5 pts) Find the inverse of the following function or explain why it is not invertible: $f : \mathbb{R} \to \mathbb{R}$:
$$f(x) = (x + 101)^{1/3}.$$

3 (10 pts.)

Prove the following statement: "a positive integer is divisible by 3 if and only if the sum of its digits are divisible by 3."

(Hint: a positive integer with digits $a_1, \ldots, a_n$ can be written as $a_1 10^{n-1} + a_2 10^{n-2} + \cdots + a_n 10^0$.)

(Hint 2: the problems on my exams are not sorted by difficulty, so you're encouraged to look ahead if you find this problem very challenging)

4 (10 pts.)

1. (5 pts) Prove or disprove: for natural numbers a, b, c, if $a|c$ and $b|c$, then $(ab)|c$.

2. (5 pts) What are the positive integers $d < 13$ such that the greatest common divisor of d and 12 equals 2?

5 (10 pts.)

1. (5 pts) Find (explain your answer) the sum

$$1/3 + 1 + 3 + 9 + 27 + \cdots + 3^{10},$$

   where each term is 3 times the previous term.

2. (5 pts) Write out a recurrence relation that produces the sequence in the previous problem $(1/3, 1, 3, 9, 27, \ldots)$.

6 (10 pts.)

Prove that for all integers $n \geq 1$,

$$1 * 2 + 3 * 4 + 5 * 6 + \cdots + (2n-1) * 2n = \frac{n(n+1)(4n-1)}{3}.$$

7 (10 pts.)

1. (5 pts) What is the Cartesian product of $\{2\}$ and $\{\{\}, 3, \{1, 2\}\}$?

2. (5 pts) How many elements are in the power set of the power set of $\{1, 2, 3\}$?

8 (10 pts.)

(a) (5 pts) Construct a relation R on $A = \{4, 5, 6\}$ that is antisymmetric and reflexive.

(b) (5 pts) Construct a relation R on $A = \{4, 5, 6\}$ that is transitive and reflexive.

9 (10 pts.)

An alien plays poker with you with her home planet's playing cards. There are 5 colors of cards, with each color having 10 cards labeled 1 through 10, for a total of 50 cards.

Each *hand* is a selection of 3 cards. The order of the cards in a hand does not matter – when we count hands, only count reorderings of the hand once.

(a) (5 pts) A hand is called *powerful* if it is all the same color. How many possible powerful hands are there?

(b) (5 pts) A hand is called *colorful* if it is all different colors. How many possible colorful hands are there?

10 (10 pts.)

(a) (5 pts) Prove or disprove: the sum of every two prime numbers is even.

(b) (5 pts) Prove or disprove: the sum of every two adjacent perfect squares is odd.