Sample Final Exam
Math 31, Spring 2010

1. Compute the following integrals. Show all your work, and do not simplify your final answers.

(a) (8 points) $\int x\cos(17x)\,dx$.

(b) (8 points) $\int_5^7 \cos x\sqrt{3+\sin x}\,dx$.

(c) (8 points) $\int \frac{e^{2x}}{e^{2x}+13}\,dx$.

2. (12 points) Determine if the following improper integral is convergent or divergent. If it is convergent, find its value (evaluate the integral).

$$\int_0^\infty \frac{x^2}{(x^3+11)^2}\,dx$$

3. (16 points)

(a) Let $z = 2-5i$ and $w = 3+4i$. Calculate $z+w$, zw, and $\frac{z}{w}$. (Do not use the polar/modulus-argument/$re^{i\theta}$ forms of z and w.) No explanation necessary, but show all your work.

(b) Let $z = -1-\sqrt{3}i$. Calculate the polar form of z (i.e., calculate r and θ such that $z = re^{i\theta}$).

4. (14 points) Determine if the series

$$\sum_{n=1}^{\infty} \frac{4^n-1}{5^n}$$

converges or diverges. Briefly **JUSTIFY** your answer.

5. (14 points) Determine if the series

$$\sum_{n=1}^{\infty} \frac{1+\sqrt{n}}{3n-2}$$

converges or diverges. Briefly **JUSTIFY** your answer.

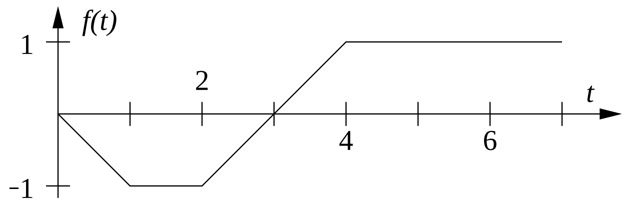
6. (16 points) Consider the solid obtained by rotating the region bounded by the curves $x=1$, $x=3$, and $y=9x-x^3$ around the x-axis.

(a) Sketch the region, the solid, and a typical disk or washer.

(b) Find the volume of the solid.

No explanation necessary, but show all your work, and do not simplify your final numerical answer.

7. (16 points) Consider the function $f(t)$ whose graph is shown below.

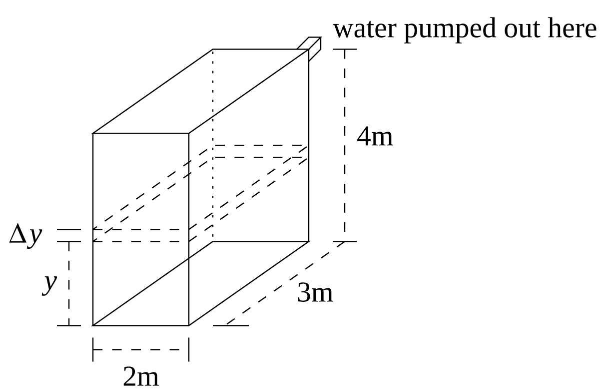


Let $g(x) = \int_0^x f(t)\, dt$.

(a) For which x $(0 \leq x \leq 7)$ is $g(x)$ increasing?

(b) What is the **maximum** value that $g(x)$ takes for $0 \leq x \leq 7$ (i.e., what is the maximum possible $y = g(x)$ for $0 \leq x \leq 7$), and at what value(s) of x is this maximum value attained? Briefly **justify** your answer.

8. (16 points) A rectangular tank is filled to the top with water, as shown below (picture not to scale).



Recall that the density of water is 1000 kg/m^3, that the acceleration due to gravity at the earth's surface is 9.8 m/s^2, the metric unit of force is the newton (N = kg$\cdot$m/s^2), and the metric unit of work is the joule (1 joule = 1 newton$\cdot$1 meter).

(a) Find the mass of a rectangular "slice" of water in the tank of thickness Δy at a height of y meters above the bottom of the tank, as indicated in the picture. (If this part doesn't make sense to you, go on to the last part.)

(b) Find the work required to move a rectangular "slice" of water in the tank of thickness Δy currently at a height of y meters to the top of the tank (height 4 meters). (If this part doesn't make sense to you, go on to the next part.)

(c) Find the total work required to pump all of the water out of an outlet at the top of the tank, as indicated. Do not simplify your final numerical answer, and clearly indicate your final answer, using the correct units.

9. (18 points) Consider the power series

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n+3}.$$

(a) Find the radius of convergence of this power series.

(b) Carefully find the interval of convergence of this power series. Make sure that you **JUSTIFY** all statements about convergence or divergence that you make in your answer.

10. (18 points) Consider the function $y = f(x)$ described by the table below.

x	1.0	1.5	2.0	2.5	3.0
$f(x)$	1.4	1.8	2.1	2.3	2.4

Suppose that $f(x)$ is continuous and **increasing** for $1 \le x \le 3$.

(a) Find the approximation L_4 (Left endpoint rule, 4 subintervals) for $\int_1^3 f(x)\,dx$. Show all your work.

(b) Is L_4 greater than or less than the actual value of $\int_1^3 f(x)\,dx$? Briefly **explain** your answer with a picture.

(c) Find the approximation T_4 (Trapezoidal rule, 4 subintervals) for $\int_1^3 f(x)\,dx$. Show all your work.

11. (18 points) Let $f(x)$ be defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n^2+1}.$$

It can be shown that the radius of convergence of this power series is 1. (In other words, you are given that the radius of convergence of the power series is 1, and you do not need to find the radius of convergence yourself.)

(a) Find a power series representation for the function $xf(2x^2)$. No explanation necessary, but show all your work.

(b) What is the radius of convergence of the power series for $xf(2x^2)$ that you found in part (a)? Briefly **justify** your answer.

(c) Use term-by-term differentiation to find a power series representation for the function $f'(x)$. No explanation necessary, but show all your work.

12. (18 points) Suppose $f(x)$ is a function such that the values of $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, and $f^{(4)}(0)$ are as shown below. Suppose also that $f(x)$ is equal to its Maclaurin series $\sum_{n=0}^{\infty} c_n x^n$ (i.e., the Taylor series of $f(x)$ centered at 0) within its interval of convergence. In other words, suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

whenever the series converges.

The goal of this problem is to calculate the Maclaurin series of $f(x)$ and put it to practical use.

(a) Fill in the missing entries $c_0, \ldots, c_4$ in the table below.

(b) Assuming the pattern continues, find a formula for c_n, the nth coefficient of the Maclaurin series of $f(x)$.

(c) Use part (a) and part (b) to find a formula for the Maclaurin series of $f(x)$. You do not need to calculate the radius of convergence of the Maclaurin series.

(d) Use the x^0 (constant) through x^4 terms of the Maclaurin series of $f(x)$ to approximate the value of $f(0.5)$. (You may assume that 0.5 lies within the interval of convergence of the Maclaurin series.) Show all your work, and round off your final answer to 3 decimal places.

$f(0) = \dfrac{1}{3^0}$	$f'(0) = \dfrac{2}{3^1}$	$f''(0) = \dfrac{6}{3^2}$	$f'''(0) = \dfrac{24}{3^3}$	$f^{(4)}(0) = \dfrac{120}{3^4}$
$c_0 =$	$c_1 =$	$c_2 =$	$c_3 =$	$c_4 =$