Sample Exam 1
Math 221A, Fall 2021

This is actually most of an undergraduate abstract algebra I final, since we have so far covered the majority of undergraduate abstract algebra I (albeit quickly). You should expect there to be fewer questions, and ones with a different emphasis (e.g., more examples from matrix groups). In other words, the problems will more resemble the ones from PS01–03, though they will not be as difficult or as long as those problems.

Fundamentally, then, this is mainly a guide to the *format* of your exam, with types of questions drawn from definitions, calculations, true/false, and proofs.

1. (15 points) Let

$$\alpha = (1\ 10\ 8)(2\ 4)(3\ 11)(5\ 12\ 7\ 6\ 9)$$
$$\beta = (2\ 7\ 3\ 9\ 6\ 10\ 12\ 4)(5\ 11\ 8)$$

Calculate $\alpha\beta$, α^{-1}, and $|\alpha|$. Put all permutation answers in cycle form, and show all your work.

For questions 2–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (13 points) **(TRUE/FALSE)** If a and b are elements of the dihedral group D_n for some $n \geq 3$, then it must be the case that $ab = ba$.

3. (13 points) **(TRUE/FALSE)** Suppose G and $\overline{G}$ are groups and $\varphi : G \to \overline{G}$ is a surjective (onto) homomorphism. If G is abelian, then it must be the case that $\overline{G}$ is abelian.

4. (13 points) **(TRUE/FALSE)** Let G be a group and let a be an element of G of finite order n. If k divides n, then it must be the case that the order of a^k is $\dfrac{n}{k}$.

5. (13 points) **(TRUE/FALSE)** Let G be a finite group and let H be a subgroup of G. If a and b are elements of G such that $aH = bH$, then it must be the case that $a = b$.

6. (16 points) **PROOF QUESTION.** Recall that $Z(G)$, the center of G, is the subgroup of all $a \in G$ such that $ax = xa$ for all $x \in G$.

Suppose G is a group, N is a normal subgroup of G, and a is an element of $Z(G)$ (the center of G). Prove that aN is an element of $Z(G/N)$ (the center of the factor group G/N).

7. (16 points) **PROOF QUESTION.** Suppose that G is a group, H is a subgroup of G, and a is a fixed element of G such that $ax = xa$ for all $x \in G$. Define

$$L = \{ha^n \mid h \in H,\ n \in \mathbf{Z}\}.$$

(In other words, L is the set of all products of an element of H and a power of a.)

Prove that L is a subgroup of G.

8. (16 points) **PROOF QUESTION.** Let G and $\overline{G}$ be groups, and let $\varphi : G \to \overline{G}$ be a homomorphism. Suppose H is a subgroup of G such that $|H| = 10$ and $a \in H$. Prove that $\varphi(a)^{10} = \overline{e}$, where $\overline{e}$ is the identity element of $\overline{G}$.

9. (16 points) **PROOF QUESTION.** Let G be a group of order 35. Prove that if G has no elements of order 7, then it must contain an element of order 5.