Math 221A, problem set 09
Due: Mon Nov 15
Last revision due: Mon Dec 06

Problems to be turned in: Problem x.y.z of Artin denotes problem y.z in Chapter x.

1. Let G be a finite simple group.
   (a) Prove that if G acts on a set of n objects (e.g., the cosets of a subgroup of index n), then G is isomorphic to a subgroup of A_n.
   (b) Prove that if G has n p-Sylow subgroups, then $|G|$ divides $n!/2$.
2. Prove that if G is a nonabelian simple group such that $|G| \leq 100$, then $|G| = 60$. Specifically, describe which problems in PS08 and PS09 and other results eliminate which orders ≤ 100.
3. Let G be a simple group of order 60.
   (a) How many elements of order 3 are there in G? Order 5? Prove your answer.
   (b) Suppose H_1 and H_2 are Sylow 2-subgroups of G, $|H_1 \cap H_2| = 2$, and $H_1 \cap H_2 = \langle x \rangle$ (i.e., H_1 and H_2 are distinct but have nontrivial intersection). Let $K = C_G(x)$. Prove that $|K|$ is a multiple of 4 and $|K| \geq 6$, and that consequently, G has a subgroup of index ≤ 5.
   (c) Now suppose $|H_1 \cap H_2| = 1$ for any two distinct 2-Sylow subgroups. Prove that there are at at most 5 2-Sylow subgroups in G.
   (d) Prove that G is isomorphic to a subgroup of A_5, and is therefore isomorphic to A_5.
4. Let G be the octahedral group (symmetry group of the octahedron). Note that we may choose the vertices of the octahedron to be the unit vectors $\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3$ in $\mathbf{R}^3$. Let $R : G \to GL_3$ be the standard representation of G as a group of orthogonal matrices of determinant 1.
   (a) Choose some $x \in G$ of order 4 and write out the matrix R_x.
   (b) Choose some $y \in G$ of order 3 and write out the matrix R_y.
   (c) Choose some $z \in G$ of order 2 corresponding to the stabilizer of an edge of the octahedron and write out the matrix R_z.
5. Artin 10.2.3(a).