Math 221A, problem set 04
Due: Wed Sep 29
Last revision due: Tue Nov 23

Supplemental definitions:

semidirect product Let N and H be groups, and let φ be a homomorphism from H to $\mathrm{Aut}(N)$, i.e., $\varphi_h : N \to N$ is an automorphism. The *semidirect product* $N \rtimes H$ is defined to be:

- *Set:* Cartesian product $N \times H$.
- *Operation:* For $n_1, n_2 \in N$, $h_1, h_2 \in H$, we define

$$(n_1, h_1)(n_2, h_2) = (n_1 \varphi_{h_1}(n_2), h_1 h_2).$$

Note that if we think of the operation of $N \rtimes H$ as declaring that

$$n_1 h_1 n_2 h_2 = n_1 \varphi_{h_1}(n_2) h_1 h_2,$$

we see that the operation is really a "move-past" rule whereby we can move h_1 past n_2 at the cost of applying the automorphism φ_{h_1}. Special case: If $\varphi_h = \mathrm{id}$ for all $h \in H$, then $N \rtimes H$ is precisely the direct product $N \times H$.

v perp For $\mathbf{v} \in \mathbf{R}^n$, we define

$$\mathbf{v}^\perp = \{x \in \mathbf{R}^n \mid \mathbf{x} \cdot \mathbf{v} = 0\}.$$

In other words, $\mathbf{v}^\perp$ is the set of all vectors of $\mathbf{R}^n$ that are orthogonal (so for nonzero $\mathbf{v}$, perpendicular) to $\mathbf{v}$ in $\mathbf{R}^n$.

reflection For nonzero $\mathbf{v} \in \mathbf{R}^n$, we define the reflection $r_\mathbf{v} : \mathbf{R}^n \to \mathbf{R}^n$ to be the linear transformation

$$r_\mathbf{v}(\mathbf{x}) = \mathbf{x} - 2\left(\frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}.$$

Problems to be turned in: Problem x.y.z of Artin denotes problem y.z in Chapter x.

1. Let N and H be group and let φ be a homomorphism from H to $\mathrm{Aut}(N)$. Prove that the semidirect product $N \rtimes H$ satisfies the axioms of a group.

2. Let $\mathbf{v}$ be a nonzero vector in $\mathbf{R}^n$.

   (a) Prove that the reflection $r_\mathbf{v}$ is an orthogonal transformation.

   (b) Prove that $r_\mathbf{v}$ has order 2.

   (c) Prove that $r_\mathbf{v}$ fixes every vector in $\mathbf{v}^\perp$.

   (d) Prove that if A is the matrix of $r_\mathbf{v}$ with respect to some basis, then $\det A = -1$. (Suggestion: $\det A$ doesn't depend on the basis you choose, so choose a basis that makes computing A simpler.)

3. We know that every element of $O_2(\mathbf{R})$ with determinant -1 has the form $R = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. Find a unit vector $\mathbf{v}$ such that $r_{\mathbf{v}}$ has matrix R with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, and prove your answer.

4. The goal of this problem is to prove the following theorem.

   Reflection Theorem: Every element of $O_n(\mathbf{R})$ is the product of at most n reflections, where the product of 0 reflections is the identity.

   In particular, $O_n(\mathbf{R})$ is generated by reflections.

   (a) Let $\mathbf{x}$ and $\mathbf{y}$ be distinct nonzero vectors in $\mathbf{R}^n$ such that $|\mathbf{x}| = |\mathbf{y}|$. Prove that there exists some vector $\mathbf{v} \in \mathbf{R}^n$ such that $r_{\mathbf{v}}(\mathbf{x}) = \mathbf{y}$. (Suggestion: Try drawing this in $\mathbf{R}^2$.)

   (b) Suppose T is a linear transformation in $O_n(\mathbf{R})$. Prove that there exists some $\mathbf{v} \in \mathbf{R}^n$ such that $(r_{\mathbf{v}} \circ T)(\mathbf{e}_n) = \mathbf{e}_n$, where $\mathbf{e}_n$ is the nth standard basis vector, and prove that the matrix of $r_{\mathbf{v}} \circ T$ with respect to the standard basis has the form $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$, where $A \in O_{n-1}(\mathbf{R})$.

   (c) Use induction on n to prove the Reflection Theorem. (What are the elements of $O_1(\mathbf{R})$?)

5. This problem gives an alternative proof of Euler's Theorem that any element of $SO_3(\mathbf{R})$ is a rotation.

   (a) Prove that a product of two reflections in $O_n(\mathbf{R})$ must fix an $(n-2)$-dimensional subspace. (Prop. 3.6.6(a) may be helpful.)

   (b) Use the Reflection Theorem to prove that if T is a linear transformation in $SO_3(\mathbf{R})$, then T is a rotation.

6. This problem expresses the isometry group of $\mathbf{R}^n$ in semidirect product form.

   (a) For an orthogonal operator φ in $O_n(\mathbf{R})$ and a translation $t_{\mathbf{v}}$, give a geometric description of $\varphi t_{\mathbf{v}} \varphi^{-1}$. (Suggestion: Compute $\varphi t_{\mathbf{v}} \varphi^{-1}(\mathbf{x})$ in terms of φ and $\mathbf{v}$.)

   (b) Prove that the isometry group G of $\mathbf{R}^n$ is isomorphic to the semidirect product $\mathbf{R}^n \rtimes O_n(\mathbf{R})$, where $\mathbf{R}^n$ is the group of translations of $\mathbf{R}^n$, and the homomorphism from $O_n(\mathbf{R})$ to $\mathrm{Aut}(\mathbf{R}^n)$ sends the matrix A to the map φ_A, where $\varphi_A(\mathbf{v}) = A\mathbf{v}$. (Suggestion: This essentially means showing that the operation in G is equivalent to the operation in $\mathbf{R}^n \rtimes O_n(\mathbf{R})$.)