More about inverse trig functions
Math 19

As we saw, the inverse trig functions are defined by:

$$\begin{aligned}
\sin^{-1} x = \theta &\quad\Leftrightarrow\quad \sin\theta = x \text{ and } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\\
\cos^{-1} x = \theta &\quad\Leftrightarrow\quad \cos\theta = x \text{ and } 0 \leq \theta \leq \pi,\\
\tan^{-1} x = \theta &\quad\Leftrightarrow\quad \tan\theta = x \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2},
\end{aligned}$$

The trick of giving $\sin^{-1} x$, $\cos^{-1} x$, or $\tan^{-1} x$ a name like θ (or α, β, etc.) is sometimes useful in calculating exact values of trig functions.

Example. Suppose we want to calculate the exact value of $\cos\left(\sin^{-1}\left(-\frac{1}{13}\right)\right)$. Let's abbreviate $\theta = \sin^{-1}\left(-\frac{1}{13}\right)$. Note that by definition, $\sin\theta = -\frac{1}{13}$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and so θ is in Quadrant IV. (More precisely, $-\frac{\pi}{2} \leq \theta \leq 0$.) In these terms, our goal is to compute

$$\cos\theta = \cos\left(\sin^{-1}\left(-\frac{1}{13}\right)\right).$$

We know that $\cos^2\theta + \sin^2\theta = 1$ (the Mother Of All Trig Identities), so we have that

$$\begin{aligned}
\cos^2\theta + \left(-\frac{1}{13}\right)^2 &= 1,\\
\cos^2\theta &= 1 - \left(-\frac{1}{13}\right)^2 = \frac{168}{169},\\
\cos\theta &= \pm\sqrt{\frac{168}{169}} = \pm\frac{\sqrt{168}}{13}.
\end{aligned}$$

However, since θ is in Quadrant IV, $\cos\theta > 0$, and the final answer is:

$$\cos\left(\sin^{-1}\left(-\frac{1}{13}\right)\right) = \cos\theta = +\frac{\sqrt{168}}{13}.$$

Problems. Find the exact values of the following expressions. Briefly justify any sign choices you make.

A. $\sin\left(\cos^{-1}\left(\frac{4}{7}\right)\right)$

B. $\tan\left(\sin^{-1}\left(-\frac{3}{8}\right)\right)$