Summary of pointwise and uniform convergence
Math 131A

Suppose we have a sequence of functions $f_n : S \to \mathbf{R}$ that converges to a function $f : S \to \mathbf{R}$ pointwise, i.e., for all $x \in S$, $\lim_{n\to\infty} f_n(x) = f(x)$. In particular, the example in which we are most interested is the example of a power series, for which we have

$$f_n(x) = \sum_{k=0}^{n} a_k x^k, \qquad f(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad S = (-R, R),$$

where R is the radius of convergence of $\sum_{k=0}^{\infty} a_k x^k$.

The following table describes how certain properties of f_n transfer, or fail to transfer, to f. Below, we assume $[a, b] \subseteq S$, and for power series, we assume $S = (-R, R)$. "Ex. n" refers to one of the examples below; other citations are from Ross.

	$f_n \to f$ pointwise	$f_n \to f$ uniformly	Power series
If the f_n are continuous on S, must f be continuous on S?	No (Ex. 1)	Yes (Thm. 24.3)	Yes (Cor. 26.2)
If the f_n and f are continuous on S, must $\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$?	No (Ex. 2)	Yes (Thm. 25.2)	Yes (Thm. 26.4)
If the f_n are differentiable on S, must f be differentiable on S?	No (Ex. 1)	No (Ex. 3)	Yes (Thm. 26.5)
If the f_n and f are differentiable on S, must $f'(x) = \lim_{n\to\infty} f_n'(x)$?	No (Ex. 4)	No (Ex. 4)	Yes (Thm. 26.5)

Example 1. Consider

$$f_n(x) = x^n, \qquad f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1, \end{cases} \qquad S = [0, 1].$$

Then for $x < 1$, $\lim_{n\to\infty} x^n = 0$, and for $x = 1$, $1^n = 1$, so $\lim_{n\to\infty} f_n(x) = f(x)$. However, $f_n(x)$ is continuous on $[0, 1]$ and $f(x)$ is not.

Example 2. Consider

$$\begin{aligned} f_n(x) &= \begin{cases} 2^{2n+2} x & \text{if } 0 \le x < \dfrac{1}{2^{n+1}}, \\ 2^{2n+2}\left(\dfrac{1}{2^n} - x\right) & \text{if } \dfrac{1}{2^{n+1}} \le x < \dfrac{1}{2^n}, \\ 0 & \text{otherwise,} \end{cases} \\ f(x) &= 0, \\ S &= [0, 1]. \end{aligned}$$

The above formulas are somewhat impenetrable, so the reader may prefer the graphs in Figure 1. The point is that the graph of each $f_n(x)$ is a trangle of area 1 (base $\frac{1}{2^n}$, height 2^{n+1}), which means that $\int_a^b f_n(x)\,dx = 1$ for all $n \in \mathbf{N}$. However, $f_n(0) = 0$, and for any $x \in (0, 1]$, if we choose N such that $\frac{1}{2^N} < x$, then for $n > N$, $f_n(x) = 0$. It follows that $\lim_{n\to\infty} f_n(x) = 0$.

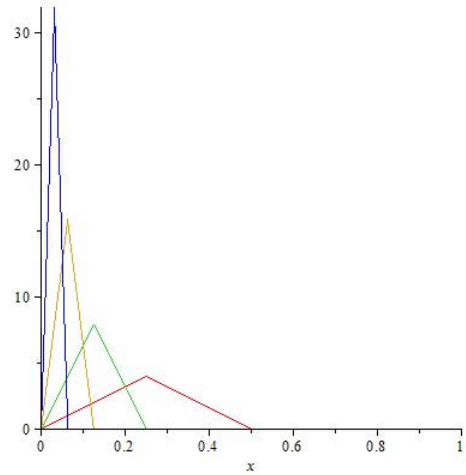

Figure 1: The witch's hat sequence

Example 3. Consider

$$f_n(x) = |x|^{1+(1/n)}, \qquad f(x) = |x|, \qquad S = [-1, 1].$$

For fixed $x \geq 0$, x^t is a continuous function of $t > 0$, so $\lim_{n\to\infty} f_n(x) = f(x)$. However, from PS08, each $f_n(x)$ is differentiable on S, but $f(x)$ is not differentiable at 0. It is nevertheless true, but harder to show, that f_n converges uniformly to f. One approach is to restrict our attention to $x > 0$ by symmetry, and let

$$D_n = \max\left\{ x - x^{1+(1/n)} \,\middle|\, x \in S \right\}.$$

For fixed n, we can then use calculus to prove that $D_n = \frac{1}{n(1 + (1/n))^{n+1}}$ (exercise), which means that $\lim_{n\to\infty} D_n = 0$ and convergence is uniform.

Example 4. Consider

$$f_n(x) = \frac{x^{n+1}}{n+1}, \qquad f(x) = 0, \qquad S = [0, 1].$$

Because $|f_n(x) - f(x)| \leq \frac{1}{n+1}$ for $x \in S$, f_n converges uniformly to f on S. However, $f_n'(x) = x^n$, so $\lim_{n\to\infty} f_n'(1) = 1 \neq 0 = f(1)$.