More about convergence and divergence tests for series
Math 131A

In these notes, we give alternate versions of some of the most useful convergence/divergence tests. We assume the comparison test as background.

Theorem 1 (Limit comparison test). *Let $\sum a_n$ and $\sum b_n$ be series, and suppose that there exist constants $K \in \mathbf{N}$ and $L, M \in \mathbf{R}$ such that for $n \geq K$, we have that $a_n, b_n > 0$ and*

$$0 < L \leq \frac{a_n}{b_n} \leq M. \tag{1}$$

Then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof. Exercise. (Apply the ordinary comparison test.) □

Corollary 2. *If there exists some $K_1 \in \mathbf{N}$ such that $a_n, b_n > 0$ for $n \geq K_1$, and also $\lim \frac{a_n}{b_n} = C$, where $0 < C < +\infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.*

Proof. Exercise. (Apply Theorem 1; see below for an example of how to show that the required hypothesis holds.) □

We also record the following simplified versions of the ratio and root tests, along with their correspondingly somewhat simpler proofs.

Theorem 3 (Simplified root test). *Let $\sum a_n$ be a series, and suppose*

$$\lim |a_n|^{1/n} = L.$$

1. *If $0 \leq L < 1$, then $\sum a_n$ converges absolutely.*
2. *If $L > 1$ (including $L = +\infty$), then $\sum a_n$ diverges.*

Proof. We first observe that by the definition of limit and the Archimedean Property, in all cases, for any $\epsilon > 0$, there exists $K \in \mathbf{N}$ such that for $n \geq K$, we have

$$\left| |a_n|^{1/n} - L \right| < \epsilon, \tag{2}$$

or in other words,

$$L - \epsilon < |a_n|^{1/n} < L + \epsilon. \tag{3}$$

In the case $L < 1$, taking $\epsilon = \frac{1-L}{2} > 0$ in (3) and letting $r = \frac{L+1}{2} < 1$, we see that there exists $K \in \mathbf{N}$ such that for $n \geq K$, we have

$$|a_n|^{1/n} < L + \epsilon = r < 1. \tag{4}$$

It follows that $|a_n| < r^n$ for $n \geq K$, so $\sum |a_n|$ converges by comparison with the convergent geometric series $\sum r^n$.

In the case $L > 1$, taking $\epsilon = \dfrac{L-1}{2} > 0$ in (3) and letting $r = \dfrac{L+1}{2} > 1$, we see that there exists $K \in \mathbf{N}$ such that for $n \geq K$, we have

$$1 < r = L - \epsilon < |a_n|^{1/n}. \tag{5}$$

It follows that $|a_n| > r^n$, and therefore, that $\lim |a_n| = +\infty$. Now, if $\lim a_n = 0$, it would follow that $\lim |a_n| = 0$; contradiction. Therefore, $\lim a_n \neq 0$, and $\sum a_n$ diverges by the nth term test. □

Theorem 4 (Simplified ratio test). *Let $\sum a_n$ be a series such that $a_n \neq 0$ for all n, and suppose*

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = L.$$

1. *If $L < 1$, then $\sum a_n$ converges absolutely.*
2. *If $L > 1$ (including $L = +\infty$), then $\sum a_n$ diverges.*

Proof. By the argument in the proof of Theorem 3, we see that for any $\epsilon > 0$, there exists $K \in \mathbf{N}$ such that for $n \geq K$,

$$L - \epsilon < \left| \frac{a_{n+1}}{a_n} \right| < L + \epsilon. \tag{6}$$

In the case $L < 1$, again following the proof of Theorem 3, we see that for $r = \dfrac{L+1}{2} < 1$, there exists $K \in \mathbf{N}$ such that for $n \geq K$, we have

$$\left| \frac{a_{n+1}}{a_n} \right| < r < 1, \tag{7}$$

or in other words, $|a_{n+1}| < r\,|a_n|$. An easy induction then shows that for $n \geq K$,

$$|a_n| \leq |a_K|\, r^{n-K}. \tag{8}$$

Therefore, making the change of variables $m = n - K$, since the geometric series

$$\sum_{n=K}^{\infty} |a_K|\, r^{n-K} = \sum_{m=0}^{\infty} |a_K|\, r^m \tag{9}$$

converges, by comparison, so does $\sum a_n$.

In the case $L > 1$, we see by Exercise 9.12 that $\lim |a_n| = +\infty$. By the argument used in the $L > 1$ case of the root theorem, it must be the case that $\lim a_n \neq 0$, so $\sum a_n$ diverges by the nth term test. □

Example 5. Problem: Let $a_n = \dfrac{7n^{25} + 11n^{12}}{3^n - 5n^{16}}$. Does $\sum a_n$ converge or diverge?

Before we start the problem, we observe that by Asymptotics, $7n^{25} >> 11n^{12}$ and $3^n >> 5n^{16}$, so we should suspect (without proof as of yet) that the problem will boil down to the convergence or divergence of $\sum \dfrac{7n^{25}}{3^n}$. So, letting $b_n = \dfrac{7n^{25}}{3^n}$ and forgetting about the original problem for the moment, here are two ways to approach $\sum b_n$.

1. Applying the Ratio Test, we see that

$$\begin{aligned}
\left|\frac{b_{n+1}}{b_n}\right| &= \frac{b_{n+1}}{b_n} \\
&= \left(\frac{7(n+1)^{25}}{3^{n+1}}\right) \Big/ \left(\frac{7n^{25}}{3^n}\right) \\
&= \left(\frac{3^n}{3^{n+1}}\right)\left(\frac{7(n+1)^{25}}{7n^{25}}\right) \\
&= \left(\frac{1}{3}\right)\left(\frac{n+1}{n}\right)^{25} \\
&= \left(\frac{1}{3}\right)\left(1+\frac{1}{n}\right)^{25}.
\end{aligned} \tag{10}$$

Therefore, by the limit laws and the fact that $\lim \frac{1}{n} = 0$,

$$\lim\left|\frac{b_{n+1}}{b_n}\right| = \lim \frac{1}{3}\left(1+\frac{1}{n}\right)^{25} = \frac{1}{3}\left(\lim\left(1+\frac{1}{n}\right)\right)^{25} = \frac{1}{3}(1^{25}) = \frac{1}{3} < 1, \tag{11}$$

which means that $\sum b_n$ converges, by the Ratio Test.

2. Applying the Root Test, we see that

$$|b_n|^{1/n} = \left(\frac{7n^{25}}{3^n}\right)^{1/n} = \frac{7^{1/n}n^{25/n}}{3} = \frac{7^{1/n}(n^{1/n})^{25}}{3}. \tag{12}$$

Therefore, by the limit laws and the fact that $\lim n^{1/n} = \lim a^{1/n} = 1$,

$$\lim |b_n|^{1/n} = \lim \frac{7^{1/n}(n^{1/n})^{25}}{3} = \frac{(\lim 7^{1/n})(\lim n^{1/n})^{25}}{3} = \frac{1}{3} < 1, \tag{13}$$

which means that $\sum b_n$ converges, by the Root Test.

Returning to the original problem, we hope that we can now compare $\sum a_n$ to $\sum b_n$ via the Limit Comparison Test. So first,

$$\begin{aligned}
\frac{a_n}{b_n} &= \left(\frac{7n^{25}+11n^{12}}{3^n-5n^{16}}\right) \Big/ \left(\frac{7n^{25}}{3^n}\right) \\
&= \left(\frac{7n^{25}+11n^{12}}{3^n-5n^{16}}\right)\left(\frac{3^n}{7n^{25}}\right) \\
&= \frac{7n^{25}(3^n)+11n^{12}(3^n)}{7n^{25}3^n-35n^{41}} \\
&= \frac{1+\left(\frac{11}{7n^{13}}\right)}{1-\left(\frac{5n^{16}}{3^n}\right)},
\end{aligned} \tag{14}$$

where in the last step, we divide top and bottom by $7n^{25}3^n$. Therefore, by Asymptotics,

$$\begin{aligned}
\lim \frac{a_n}{b_n} &= \lim \frac{1 + \left(\frac{11}{7n^{13}}\right)}{1 - \left(\frac{5n^{16}}{3^n}\right)}, \\
&= \frac{1 + \lim \left(\frac{11}{7n^{13}}\right)}{1 - \lim \left(\frac{5n^{16}}{3^n}\right)}, \\
&= \frac{1+0}{1-0} = 1.
\end{aligned} \tag{15}$$

Since $0 < 1 < +\infty$, by the Limit Comparison Test, $\sum a_n$ converges if and only if $\sum b_n$ converges. However, since we already showed that $\sum b_n$ converges, $\sum a_n$ converges as well.