Absolute/conditional convergence and rearrangements
Math 131A

In these notes, we examine the question: When does the order of summation affect the convergence or divergence of a series? For convenience, we assume that (after renumbering) the domain of every sequence is $\mathbf{N}$.

Definition 1. A *rearrangement* of a sequence (a_n) is a sequence (b_n) such that

$$b_n = a_{\sigma(n)} \tag{1}$$

for some bijection $\sigma : \mathbf{N} \to \mathbf{N}$. Similarly, if (b_n) is a rearrangment of (a_n), we also say that $\sum b_n$ is a rearrangement of $\sum a_n$.

Theorem 2. *Let (a_n) be a sequence such that $a_n \geq 0$ for $n \in \mathbf{N}$, and let (b_n) be a rearrangement of (a_n). If $\sum a_n$ converges, then $\sum b_n$ converges.*

Proof. Suppose $b_n = a_{\sigma(n)}$ for some bijection $\sigma : \mathbf{N} \to \mathbf{N}$. By the Cauchy Criterion, we know that for any $\epsilon > 0$, there exists some $N_a(\epsilon)$ such that if $m > k > N_a(\epsilon)$, then

$$\left| \sum_{n=k}^{m} a_n \right| < \epsilon. \tag{2}$$

So now, for $\epsilon > 0$, let

$$S(\epsilon) = \{ n \in \mathbf{N} \mid \sigma(n) \leq N_a(\epsilon) \}. \tag{3}$$

Since σ is a bijection, $S(\epsilon)$ is finite, so we may define $N(\epsilon) = \max S(\epsilon)$. Now suppose $m > k > N(\epsilon)$. Let

$$T = \{ \sigma(n) \mid k \leq n \leq m \}, \tag{4}$$

Since σ is a bijection, it maps the indices $k, k+1, \ldots, m$ injectively into T, which is contained (possibly properly) in the set $\{n' \mid \min T \leq n' \leq \max T\}$. Therefore, since the a_n are all nonnegative, we see that

$$\sum_{n=k}^{m} b_n = \sum_{n=k}^{m} a_{\sigma(n)} \leq \sum_{n'=\min T}^{\max T} a_{n'}. \tag{5}$$

However, since $n > \max S(\epsilon)$ for $k \leq n \leq m$, by definition of $S(\epsilon)$, we see that

$$N_a(\epsilon) < \min T \leq \max T. \tag{6}$$

Therefore, by (2),

$$\sum_{n=k}^{m} b_n \leq \sum_{n'=\min T}^{\max T} a_{n'} < \epsilon. \tag{7}$$

The theorem follows by the Cauchy Criterion. □

Corollary 3. *Any rearrangement $\sum b_n$ of an absolutely convergent series $\sum a_n$ also converges absolutely.*

Proof. If $\sum a_n$ converges absolutely, then $\sum |b_n|$ converges because it is a rearrangement of the convergent nonnegative series $\sum |a_n|$. Therefore, $\sum b_n$ converges absolutely. □

If $\sum a_n$ converges conditionally, then rearrangements are completely unpredictable. To be precise, we have the following remarkable result, due to Riemann.

Theorem 4 (Riemann rearrangement theorem). *If $\sum a_n$ converges conditionally, then for any $L \in \mathbf{R} \cup \{+\infty, -\infty\}$, there is a rearrangement of $\sum a_n$ that converges to L.*

Sketch of proof. For simplicity, assume a_n is never 0. Let $\sum b_n$ contain the positive terms of $\sum a_n$, and let $\sum c_n$ contain the negative terms. If both $\sum b_n$ and $\sum c_n$ converge, we would have

$$\sum |a_n| = \sum b_n + \left|\sum c_n\right|, \tag{8}$$

and $\sum a_n$ would converge absolutely. Furthermore, if $\sum b_n = +\infty$ and $\sum c_n$ is finite, then $\sum a_n$ would diverge, and similarly for the case where $\sum b_n$ is finite and $\sum c_n = -\infty$. Therefore, it must be that $\sum b_n = +\infty$ and $\sum c_n = -\infty$.

Note that since $\sum a_n$ converges conditionally, $a_n \to 0$, which means that $b_n, c_n \to 0$. In particular, every subset of the b_n or c_n has a largest size element (i.e., element with largest possible absolute value). So now rearrange the b_n and c_n so they are both in decreasing order of size, and without loss of generality, assume (by symmetry) that $L \geq 0$. If $L < +\infty$, we arrange $\sum a_n$ as follows:

1. Begin with the minimum number of positive terms b_n required to achieve a sum greater than L.

2. Then add the minimum number of negative terms c_n required to bring the partial sum back down to less than L.

3. Keep alternating: Add positive terms until we "overshoot" L, add negative terms until we "undershoot" L, and so on.

It can then be shown that this arrangement has a sum that converges to L.

Similarly, if $L = +\infty$, we arrange $\sum a_n$ as follows:

1. Begin with the minimum number of positive terms b_n required to achieve a sum greater than $|c_1| + 1$.

2. Then add the negative term c_1, giving a total greater than 1.

3. Keep alternating: Add new positive terms to achieve an additional sum greater than $|c_2| + 1$, then add c_2; add a sum greater than $|c_3| + 1$, then add c_3; and so on.

Again, it can then be shown that the sum of this arrangement approaches $+\infty$. □