Sample Final Exam
Math 131A, Spring 2024

In this exam, you may take the following as given:

> **Theorem:** For any $c \in \mathbf{R}$, there exists a sequence x_n in $\mathbf{Q}$ and a sequence y_n in $\mathbf{R} \backslash \mathbf{Q}$ (i.e., each x_n is rational and each y_n is irrational) such that $\lim x_n = c$ and $\lim y_n = c$.

1. (16 points) State both Fundamental Theorems of Calculus, one of which has to do with the derivative of an integral, and the other of which has to do with the integral of a derivative. For simplicity, you may assume that all functions involved in your statements are continuous.

2. (15 points)

(a) Let (a_n) be a sequence. Define what it means to be a subsequence of (a_n).

(b) State the Bolzano-Weierstrass Theorem.

3. (15 points) Let $a_n = \dfrac{n(3^{1/n})}{5n^2+7}$. Determine if the series $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n(3^{1/n})}{5n^2+7}$ converges or diverges, and prove your answer.

For questions 4–9, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

4. (13 points) **TRUE/FALSE:** Let $f_n : [0,1] \to \mathbf{R}$ be a sequence of continuous functions with domain $[0,1]$, and suppose that f_n converges to $f : [0,1] \to \mathbf{R}$ pointwise. Then it must be the case that f is continuous on $[0,1]$.

5. (13 points) **TRUE/FALSE:** Let S be a nonempty bounded subset of $\mathbf{R}$. Suppose $u \in \mathbf{R}$ satisfies the conditions that for all $s \in S$, $s \leq u$; and that if $s \leq v$ for all $s \in S$, then $u \leq v$. Then it is possible that $u \notin S$.

6. (13 points) **TRUE/FALSE:** If $g : \mathbf{R} \to \mathbf{R}$ is given by

$$g(x) = \begin{cases} \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ -1 & \text{if } x = 0, \end{cases}$$

then g is continuous at $x = 0$.

7. (13 points) **TRUE/FALSE:** Suppose $h : [-3,5] \to \mathbf{R}$ is a function such that if $a \in [-3,5]$, then $\displaystyle\lim_{x \to a} \frac{h(x) - h(a)}{x - a}$ exists. Then it must be the case that $\displaystyle\int_{-2}^{4} h(x)\, dx$ exists.

8. (13 points) **TRUE/FALSE:** Let b_n be a sequence such that $5 \leq b_n \leq 8$ for all $n \in \mathbf{N}$. Then it must be the case that $\displaystyle\lim_{n \to \infty} b_n$ exists.

9. (13 points) **TRUE/FALSE:** Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that $\lim_{x \to 4} f(x) = -7$. Then it is possible that there exists a sequence a_n such that $\lim_{n \to \infty} a_n = 4$, $a_n \neq 4$ (for all $n \in \mathbf{N}$), and $\lim_{n \to \infty} f(a_n) = 12$.

10. (15 points) **PROOF QUESTION.** Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x^3}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f is differentiable at 0.

11. (15 points) **PROOF QUESTION.** Define $g : \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \begin{cases} |x| & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Use the ϵ-δ definition of continuity to prove that g is continuous at 0.

12. (15 points) **PROOF QUESTION.**

(a) State the Mean Value Theorem for a function $f(x)$ on $[a, b]$.

(b) Recall that $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$. Use the Mean Value Theorem to prove that for any $a, b \in \mathbf{R}$, $a < b$, we have that

$$|\sin b - \sin a| \leq |b - a|.$$

13. (15 points) **PROOF QUESTION.** Let a_n be a sequence such that $0 \leq a_n \leq \dfrac{3}{n}$ for all $n \in \mathbf{N}$. Use the **definition** of the limit of a sequence (and not, for example, the Squeeze Lemma) to prove that $\lim_{n \to \infty} a_n = 0$.

14. (16 points) **PROOF QUESTION.** Let $g(x)$ and $h(x)$ be the power series defined by

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \qquad h(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

(a) Use the Ratio Test directly, and not the radius of convergence theorem, to prove that $g(x)$ converges for all $x \in \mathbf{R}$.

(b) Prove that the radius of convergence of $g(x)$ is ∞. (Suggestion: Assume that the radius of convergence of $g(x)$ is a finite number R, and obtain a contradiction.)

(c) Now assume that the radius of convergence of both $g(x)$ and $h(x)$ is ∞. Prove $h'(x) = g(x)$, justifying term-by-term operations carefully.