Math 131A, problem set 11
Outline due: Fri May 10
Completed version due: Tue May 14
Last revision due: TBA

Problems to be done but not turned in: 25.1, 25.3, 25.5, 25.7, 25.9, 25.11, 25.13, 25.15, 26.1, 26.3, 26.5, 26.7.

Problems to be turned in: All numbers refer to exercises in Ross.

1. Define $g : I \to \mathbf{R}$ by $g(x) = \displaystyle\sum_{n=1}^{\infty} \frac{3^n x^n}{n^{3/2}}$, where I is the interval of convergence of the series.
   (a) Compute the radius of convergence of g, and compute I (i.e., what happens at the boundary?), with proof.
   (b) Prove that g is continuous on I. (Suggestion: Weierstrass M-test.)
2. Ex. 25.10(a,b).
3. Define $f : (-R, R) \to \mathbf{R}$ by $f(x) = \displaystyle\sum_{n=1}^{\infty} n^2 x^n$, where R is the interval of convergence of the series.
   (a) Compute R, with proof.
   (b) Find a closed (non-series) formula for $f(x)$, with proof.
   (c) Find the exact value of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{7^n}$, with proof.
4. Define a function $E : \mathbf{R} \to \mathbf{R}$ by $E(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. (Note that by Example 1, section 23, the radius of convergence of $E(x)$ is $+\infty$.)
   (a) Prove that for all $x \in \mathbf{R}$, $E'(x) = E(x)$. Make sure to justify any term-by-term operations carefully.
   (b) Prove that for $x > 0$, $E(x) > 1$.
   (c) For $a \in \mathbf{R}$, define $f_a(x) = \dfrac{E(x+a)}{E(x)}$. Prove that for $x \geq 0$, $f_a(x) = E(a)$. (I.e., prove that for $x \geq 0$, $f_a(x)$ is the constant function $E(a)$).
   (d) Prove that for $x < 0$, $0 < E(x) < 1$.

   Remark: We usually write $E(x)$ as e^x, as in that notation, the above results show that e^x satisfies the usual properties of the exponential function. In other words, this problem shows that we may therefore take the series expansion of $E(x)$ to be the *definition* of e^x.