Math 131A, problem set 10
Outline due: Fri May 03
Completed version due: Tue May 07
Last revision due: TBA

Problems to be done but not turned in: 23.1, 23.3, 23.5, 23.7, 23.9, 24.1, 24.3, 24.5, 24.7, 24.9, 24.11, 24.13, 24.15, 24.17, and:

- Let $S \subseteq \mathbf{R}$ be nonempty, and let $f, f_n : S \to \mathbf{R}$ be functions. For $n \in \mathbf{N}$, let $D_n = \sup\{|f(x) - f_n(x)| \mid x \in S\}$. Prove that the following are equivalent (if and only if):
  - f_n converges uniformly to f on S.
  - $\lim_{n\to\infty} D_n = 0$.

Problems to be turned in: All numbers refer to exercises in Ross.

1. Let $F : [a, b] \to \mathbf{R}$ be differentiable, and suppose that $\dfrac{dF}{dx}$ is continuous on $[a, b]$. This problem gives an alternative proof of FTC $\displaystyle\int \frac{d}{dx}$ (the equation below), based on FTC $\displaystyle\frac{d}{dx}\int$. (I.e., assume Thm. 34.3 but not Thm. 34.1.)
   (a) Let $G(x) = \displaystyle\int_a^x F'(t)\,dt$, and let $H(x) = F(x) - G(x)$. Find the value of $H'(x)$. What conclusion can you draw, and why?
   (b) Prove that
   $$F(b) - F(a) = \int_a^b \frac{dF}{dx}\,dx.$$
   Suggestion: What is $H(a)$?

2. Let
   $$f(x) = 100x(x-1)(x-2)e^{-x^6}.$$
   Numerical integration shows that (rounded off)
   $$\int_0^1 f(x)\,dx \approx 23.56, \qquad \int_1^2 f(x)\,dx \approx -0.32, \qquad 0 < \int_2^\infty f(x)\,dx < 10^{-7},$$
   where by definition, $\displaystyle\int_2^\infty f(x)\,dx = \lim_{b\to\infty}\int_2^b f(x)\,dx$.
   Let $F : [0, \infty) \to \mathbf{R}$ be a differentiable function such that $F'(x) = f(x)$ and $F(0) = 13$.
   (a) Does $F(x)$ attain an absolute maximum value? If so, for which value(s) of x? Prove your answer.
   (b) Does $F(x)$ attain an absolute minimum value? If so, for which value(s) of x? Prove your answer.

   Suggestion: Do **not** try to compute a formula for the indefinite integral of $f(x)$, as it can be shown that there is no such formula expressed in terms of elementary functions.

   (Cont. on other side.)

3. Let $f(x) = \sum_{n=1}^{\infty} \frac{(-3)^n x^n}{5^n \sqrt{n}}$.

   (a) Find the radius of convergence of $f(x)$.

   (b) Find the exact interval of convergence of $f(x)$. (I.e., what happens on the boundary?)

4. Let

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

   (a) Prove that $S(x)$ and $C(x)$ converge absolutely for all $x \in \mathbf{R}$.

   (b) Explain why that implies that the radii of convergence of $S(x)$ and $C(x)$ are both equal to ∞. Warning: Keep Example 6 of Sect. 23 (p. 190) in mind.

5. Ex. 23.8.

6. Ex. 24.6.

7. Let $f_n(x) = \sum_{k=0}^{n} x^k$.

   (a) Does the sequence (f_n) converge pointwise on the set $(0, 1)$? If so, give the limit function. (Suggestion: See p. 96.)

   (b) Does (f_n) converge uniformly on $(0, 1)$? Prove your assertion. (Suggestion: Use Remark 24.4, or see the problems to be done but not turned in.)