Math 131A, problem set 08
Outline due: Wed Apr 10
Completed version due: Mon Apr 15
Last revision due: TBA

Problems to be done but not turned in: 20.13, 20.15, 20.17, 20.19; 28.1, 28.3, 28.5, 28.7, 28.9, 28.11, 28.13, 28.15. Also, learn the following:

Local Linearity Lemma. Let I be an interval and let $f : I \to \mathbf{R}$ be a function. For $a \in I$, the following are equivalent:

- f is differentiable at a and there exists some $m \in \mathbf{R}$ such that $m = f'(a)$.
- There exists $m \in \mathbf{R}$ and a function $\epsilon(x)$ (the *slope error* in the local linear approximation to f at a) such that

$$f(x) - f(a) = (x - a)(m + \epsilon(x))$$

  and $\lim_{x \to a} \epsilon(x) = 0 = \epsilon(a)$ (i.e., $\epsilon(x)$ is continuous at a and $\epsilon(a) = 0$).

(Note that as a consequence, if either condition is true, and therefore, both conditions are true, then $m = f'(a)$.)

Problems to be turned in: All numbers refer to exercises in Ross.

1. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbf{Q}, \\ 0 & \text{for } x \notin \mathbf{Q}. \end{cases}$$

   Prove that for any $c \in \mathbf{R}$, $\lim_{x \to c} f(x)$ does not exist.

2. Let $g : \mathbf{R} \setminus \{3\} \to \mathbf{R}$ be defined by $g(x) = \dfrac{1}{\sqrt[3]{x-3}}$. Prove that $\lim_{x \to 3^+} g(x) = +\infty$.

3. Let $f(x) = x^{2/3}$. Use the limit definition of derivative to calculate $f'(7)$, with proof. (Suggestion: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.)

4. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} |x|^{3/2} - 5x & \text{if } x \notin \mathbf{Q}, \\ -5x & \text{if } x \in \mathbf{Q}. \end{cases}$$

   Use the limit definition of derivative to prove that f is differentiable at $x = 0$.

5. Let $g : \mathbf{R} \to \mathbf{R}$ be defined by

$$g(x) = \begin{cases} 2x & \text{if } x \notin \mathbf{Q}, \\ 0 & \text{if } x \in \mathbf{Q}. \end{cases}$$

   Is g is differentiable at $x = 0$? Prove or disprove.

6. Let $r \in \mathbf{R}$ be fixed, and let $k_r : \mathbf{R} \to \mathbf{R}$ be defined by

$$k_r(x) = \begin{cases} |x|^r \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Determine and prove the best possible theorem of the following form:

- $\lim_{x \to 0} k_r(x) = 0$ if and only if $r >$??.

(b) Determine and prove the best possible theorem of the following form:

- $k_r(x)$ is differentiable at 0 if and only if $r >$??.

7. Let $h : \mathbf{R} \to \mathbf{R}$ be defined by

$$h(x) = \begin{cases} 2x^2 \sin\left(\dfrac{1}{x}\right) + x & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

(a) Prove that $h'(0) = 1$.

(b) Use the laws of differentiation to calculate $h'(a)$ for $a \neq 0$.

(c) Prove that for any $\epsilon > 0$, there exists some $a \in \mathbf{R}$ such that $0 < a < \epsilon$ and $h'(a) < 0$.