Math 131A, problem set 07
Outline due: Mon Mar 25
Completed version due: Fri Mar 29
Last revision due: Fri May 03

Problems to be done but not turned in: 18.1, 18.3, 18.5, 18.7, 18.9, 18.11; 19.1, 19.3; 20.1, 20.3, 20.5, 20.7, 20.9, 20.11.

Problems to be turned in: All numbers refer to exercises in Ross.

1. (a) Does there exist a continuous **onto** function $f : (0,1) \to [0,1]$? Give an example or prove that no such function exists.

   (b) Does there exist a continuous **onto** function $f : [0,1] \to (0,1)$? Give an example or prove that no such function exists.

2. Recall (PS05 problem 1) that to say that a sequence (a_n, b_n) in $\mathbf{R}^2$ converges to a limit $(L, M) \in \mathbf{R}^2$ means that $\lim a_n = L$ and $\lim b_n = M$. For $A \subseteq \mathbf{R}^2$, to say that $f : A \to \mathbf{R}$ is *continuous* means that if (x_n, y_n) is a sequence in A such that $\lim(x_n, y_n) = (a, b) \in A$, then $\lim f(x_n, y_n) = f(a, b)$.

   Prove the following theorem (XVT in $\mathbf{R}^2$, minimum half): For $a_1 \leq a_2$ and $b_1 \leq b_2$, if $A = [a_1, a_2] \times [b_1, b_2]$ and $f : A \to \mathbf{R}$ is continuous, then there exists $(c, d) \in A$ such that $f(c, d) \leq f(x, y)$ for all $(x, y) \in A$. (Suggestion: See PS05 problem 1.)

3. (a) Let $f : [0,1] \to \mathbf{R}$ and $g : [0,1] \to \mathbf{R}$ be continuous functions such that $f(0) = g(1) = 0$ and $f(1) = g(0) = 1$. Prove that there exists some $x \in [0,1]$ such that $f(x) = g(x)$.

   (b) On March 14, Molly and Emmett are both on Mt. Shasta. Molly begins her day by departing from the base of the mountain at 5am, and ends the day at 7pm at the summit of the mountain. Emmett wakes up at 8am at the summit, climbs down, and reaches the base of the mountain at 4pm.

   Prove that at some point in time on March 14, Molly and Emmett are at exactly the same elevation. You may make reasonable assumptions about continuity, but clearly state what those assumptions are.

4. Ex. 19.2(b).

5. Let $f(x) = x^3$. Carefully use the laws for function limits to evaluate $\displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$. In particular, indicate clearly where (if anywhere) you use $h \neq 0$.

6. Prove that $\displaystyle\lim_{x\to 0} \frac{1}{\sqrt{|x|}} \sin\left(\frac{1}{x}\right)$ does not exist.

7. Let $h : \mathbf{R} \to \mathbf{R}$ be defined by

$$h(x) = \begin{cases} x^2 & \text{for } x \notin \mathbf{Q}, \\ 4x - 3 & \text{for } x \in \mathbf{Q} \text{ and } x \neq 3, \\ 17 & \text{for } x = 3. \end{cases}$$

   Determine (guess) the value of $\displaystyle\lim_{x\to 3} h(x)$, and prove your answer.