Math 131A, problem set 06
Outline due: Wed Mar 13
Completed version due: Mon Mar 18
Last revision due: Fri May 03

Problems to be done but not turned in: 15.1, 15.3, 15.5, 15.7; 17.1, 17.3, 17.5, 17.7, 17.9, 17.11, 17.13, 17.15.

Problems to be turned in: All numbers refer to exercises in Ross.

1. Ex. 15.4(a,c).

2. Ex. 15.6.

3. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \sqrt[5]{x}$. Use the ϵ-δ definition of continuity to prove that f is continuous at 0.

4. For a nonempty $A \subseteq \mathbf{R}$ and $K > 0$, we say that a function $g : A \to \mathbf{R}$ is *K-Lipschitz* if for any $x, y \in A$, we have

$$|g(x) - g(y)| \leq K\,|x - y|\,.$$

   Let $A \subseteq \mathbf{R}$ be nonempty and $K > 0$, and suppose that $g : A \to \mathbf{R}$ is K-Lipschitz. Prove that g is continuous on A (i.e., at every $a \in A$).

5. Ex. 17.10(a,b).

6. Let $h : \mathbf{R} \to \mathbf{R}$ be defined by

$$h(x) = \begin{cases} (\sqrt[3]{x-2}) \sin\left(\dfrac{1}{x-2}\right) & \text{if } x \neq 2, \\ 0 & \text{if } x = 2. \end{cases}$$

   Prove or disprove: h is continuous at 2.

7. Define $f : [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x = p/q,\ p, q \in \mathbf{Z},\ q > 0,\ \gcd(p,q) = 1; \\ 0 & \text{if } x \text{ irrational.} \end{cases}$$

   (a) Prove that if $a \in [0,1]$ is rational, then f is discontinuous at a. (See the hint for Ex. 17.13; note that $f(0) = 1$.)

   (b) Prove that if $a \in [0,1]$ is irrational, then f is continuous at a. (Suggestion: For which x is $f(x) \geq 1/n$?)