Math 131A, problem set 05
CORRECTED AGAIN FRI MAR 08
Outline due: Wed Mar 06
Completed version due: Mon Mar 11
Last revision due: Fri May 03

Problems to be done but not turned in: 12.1, 12.3, 12.5, 12.7, 12.9, 12.11, 12.13; 14.1, 14.3, 14.5, 14.7, 14.9, 14.11, 14.13.

Problems to be turned in: All numbers refer to exercises in Ross. In problems 3–7, you may apply the limit laws for sequences in "outside to inside" order (i.e., calculus-style), but you must apply the series convergence laws in logical order.

1. We say that a sequence (a_n, b_n) in $\mathbf{R}^2$ *converges* to a limit $(L, M) \in \mathbf{R}^2$ if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. We also say that (a_n, b_n) is *bounded* if both (a_n) and (b_n) are bounded.

   Prove that every bounded sequence in $\mathbf{R}^2$ has a convergent subsequence. (Suggestion: Use Bolzano-Weierstrass twice, carefully.)

2. This problem proves the *Limit Comparison Test*, which is useful for the rest of this problem set. Let $\sum a_n$ and $\sum b_n$ be series with $a_n, b_n > 0$.

   (a) Suppose there exist $K, L, M \in \mathbf{R}$ such that for $n \geq K$, we have that
   $$0 < L \leq \frac{a_n}{b_n} \leq M.$$
   Prove that $\sum a_n$ converges if and only if $\sum b_n$ converges. (Suggestion: Comparison.)

   (b) Now suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = C$, where $0 < C < +\infty$. Prove that $\sum a_n$ converges if and only if $\sum b_n$ converges.

3. Determine if the series $\sum 2^{1/n} \left(\frac{13n^6 - 5n^4}{n^7 - 13n} \right)$ converges or diverges, and prove your answer (i.e., carefully apply series tests).

4. Determine if the series $\sum \frac{(n!)(2n)!}{(3n)!}$ converges or diverges, and prove your answer (i.e., carefully apply series tests).

5. Determine if the series $\sum n^{1/n} \cos(n)$ converges or diverges, and prove your answer (i.e., carefully apply series tests).

6. Determine if the series $\sum (5 + 3(-1)^n) \left(\frac{2^n n^3}{3^n - 5} \right)$ converges or diverges, and prove your answer (i.e., carefully apply series tests).

7. Determine if the series $\sum (-1)^n \left(\frac{3n^2 + 2n + 14}{5n^{7/2} - 7} \right)$ converges or diverges, and prove your answer (i.e., carefully apply series tests).