Math 131A, problem set 04
Outline due: Wed Feb 28
Completed version due: Mon Mar 05
Last revision due: Wed Mar 21

Additional definition: For $k \in \mathbf{N}$, the k-*tail* of a sequence $(s_n)_{n\in\mathbf{N}}$ is the sequence $(s_n)_{n\geq k}$, i.e., the sequence starting at k instead of 1.

Problems to be done but not turned in: 10.1, 10.3, 10.5, 10.7, 10.9, 10.11, 11.1, 11.3, 11.5, 11.7, 11.9, 11.11; and:

- For $k \in \mathbf{N}$, prove that a sequence (s_n) converges if and only the k-tail of (s_n) converges.
- This problem goes through some of the details of the definitions of lim inf and lim sup. (The answers are essentially in Sect. 10.) Let L, M be real numbers, and let s_n be a sequence such that $L \leq s_n \leq M$ for all n. Define:

$$S_k = \{s_n \mid n \geq k\}$$
$$M_k = \sup S_k,$$
$$L_k = \inf S_k.$$

  - For $k \in \mathbf{N}$, is $S_k \subseteq S_{k+1}$ or $S_{k+1} \subseteq S_k$? Prove your answer.
  - For $k \in \mathbf{N}$, is $M_k \geq M_{k+1}$ or $M_k \geq M_{k+1}$? Prove your answer. State (without proof) the analogous result for L_k.
  - Prove that for $k \in \mathbf{N}$, $L_k \leq M$. State (without proof) the analogous fact for M_k.
  - Prove that $\lim\limits_{k\to\infty} M_k$ and $\lim\limits_{k\to\infty} L_k$ both exist. Explain why this means that $\limsup s_n$ and $\liminf s_n$ are well-defined.

Problems to be turned in: All numbers refer to exercises in Ross.

1. Ex. 9.12(a).
2. Ex. 10.2.
3. Let p_n be the nth prime number (so $p_1 = 2$, $p_4 = 7$, and so on). Define a sequence s_n inductively by

$$s_1 = 13, \qquad s_{n+1} = \left(1 - \frac{1}{p_n}\right) s_n.$$

   Prove that $\lim s_n$ exists.
4. Ex. 11.2(a,c,d,e).

(cont. on next page)

5. Let S be a nonempty subset of $\mathbf{R}$. We say that $x \in \mathbf{R}$ is a *limit point* of S if for any $\epsilon > 0$, there exists some $s \in S$ such that $s \neq x$ and $|x - s| < \epsilon$.

   Prove that the following are equivalent:

   - $x \in \mathbf{R}$ is a limit point of S.
   - There exists a sequence s_n such that for each n, $s_n \in S$ and $s_n \neq x$, and $\lim s_n = x$.

   (Note that a limit point of S need not be an element of S.)

6. Let S be a bounded subset of $\mathbf{R}$ that contains infinitely many points.

   (a) Explain why there exists a sequence s_n in S such that $s_k \neq s_n$ for $k \neq n$. (Suggestion: Choose $s_1 \in S$, $s_2 \neq s_1$, $s_3 \neq s_1, s_2$, ... How can we be sure that we can always continue this process?)

   (b) Prove that S has at least one limit point in $\mathbf{R}$.

7. Let L, M be real numbers, and let s_n and t_n be sequences such that $L \leq s_n \leq M$ and $L \leq t_n \leq M$ for all n. Define:

$$\begin{aligned} S_k &= \{s_n \mid n \geq k\}, & M_{S,k} &= \sup S_k, \\ T_k &= \{t_n \mid n \geq k\}, & M_{T,k} &= \sup T_k, \\ U_k &= \{s_n + t_n \mid n \geq k\}, & M_{U,k} &= \sup U_k. \end{aligned}$$

   (a) What is the standard name of $\lim\limits_{k \to \infty} M_{S,k}$?

   (b) Prove that for all k, $M_{U,k} \leq M_{S,k} + M_{T,k}$. (Suggestion: $M_{U,k}$ is the **least** upper bound of U_k.)

   (c) Prove that $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$.