Math 131A, problem set 03
Outline due: Wed Feb 14
Completed version due: Mon Feb 19
Last revision due: Thu Mar 28

Problems to be done but not turned in: 8.1, 8.3, 8.5, 8.7, 8.9, 9.1, 9.3, 9.5, 9.7, 9.9, 9.11, 9.13, 9.15, 9.17.

Problems to be turned in: All numbers refer to exercises in Ross. In Problems 1–4, prove your results using the definition of the limit, and not the limit laws (e.g., not the Squeeze Lemma).

1. Ex. 8.4. Note that we cannot assume that $\lim_{n\to\infty} t_n$ exists.

2. (a) Give an example of convergent sequences s_n and t_n such that $s_n < t_n$ for all n, but $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$.
   (b) Prove that if s_n and t_n are convergent sequences such that $s_n \le t_n$ for all n, then $\lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n$. (Suggestion: Contradiction.)

3. (a) For $a \in \mathbf{R}$, suppose (x_n) is a sequence such that for all n, $|x_n - a| < \dfrac{1}{n}$. Prove that $\lim_{n\to\infty} x_n = a$.
   (b) Recall that to say that S is *dense* in $\mathbf{R}$ means that for any $a, b \in \mathbf{R}$ such that $a < b$, there exists some $x \in S$ such that $a < x < b$. Prove that if S is dense in $\mathbf{R}$ and $a \in \mathbf{R}$, there exists a sequence (x_n) in S such that $a < x_n$ for all n and $\lim_{n\to\infty} x_n = a$. (Suggestion: Use the previous part of the problem to choose the sequence (x_n) nonconstructively.)

4. Let a_n be a sequence such that $\lim_{n\to\infty} na_n = 2$.
   (a) Prove that there exists some K such that if $n > K$, then $\dfrac{1}{n} \le a_n \le \dfrac{3}{n}$.
   (b) Prove that $\lim_{n\to\infty} a_n = 0$.

5. Find the value of $\lim_{n\to\infty} \dfrac{3 + 2\cos n^2}{n}$, and prove your answer, using either the definition of limit or the Squeeze Lemma (Ex. 8.5).

6. Suppose that (a_n) and (b_n) are sequences such that $\lim_{n\to\infty} a_n = -3$ and $\lim_{n\to\infty} b_n = 5$. Determine the value of $\lim_{n\to\infty} \dfrac{2a_nb_n - (b_n + 1)\sqrt{7 + a_n^2}}{a_n^2 + 3}$, and carefully use the limit laws of Ch. 9 and Example 5 of Ch. 8 to prove your answer.

7. Ex. 8.8(c). Suggestion: Use the square root techniques of Ch. 8 and the limit laws of Ch. 9.