Sample Exam 3
Math 129B, Spring 2012

This exam covered through section 8.2 of the text, and not 8.3. Otherwise, it covers roughly the same material as ours will. Our exam may also be a bit longer (this was a 50 minute exam).

1. (16 points) Let $T : V \to V$ be linear, and let λ be a real number. Define the eigenspace $E_T(\lambda)$, and explain in **one** sentence what $E_T(\lambda)$ has to do with the λ-eigenvectors of T (i.e., with the eigenvectors of T with eigenvalue λ).

2. (14 points) Let A be an $n \times n$ matrix; let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$; and let $B = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for $\mathbb{R}^n$. Write down a formula for $[T]_{B,B}$ (the matrix of T relative to the basis B) in terms of A and $\mathbf{u}_1, \dots \mathbf{u}_n$.

For questions 3–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer as specifically as possible. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (12 points) It is possible that there exists a **one-to-one** linear function $L : \mathbb{P}_{11} \to \mathbb{R}^{15}$ such that $\operatorname{rank} L = 12$.

4. (12 points) It is possible that there exists a linear function $S : \mathbb{M}(2,2) \to \mathbb{P}_3$ such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in \operatorname{im} S$.

5. (12 points) Let $V = \operatorname{span}\{\cos x, \sin x, e^x\} \subseteq \mathbb{F}(\mathbb{R})$. You may take it as given that $B = \{\cos x, \sin x, e^x\}$ is a basis for V. Also, let $B' = \left\{1, x, x^2, x^3\right\}$ be the standard basis for $\mathbb{P}_3$, and let $T : V \to \mathbb{P}_3$ be a linear function such that

$$[T]_{B,B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & -3 \end{bmatrix}.$$

It is possible that $\cos x + 2\sin x$ is an element of $\ker T$.

6. (16 points) **PROOF QUESTION.** Let W be a vector space, let $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be a linearly independent subset of W, and let $L : \mathbb{R}^3 \to W$ be defined by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3$ for all $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$. You may take it as given that L is linear. Prove that L is one-to-one.

7. (18 points) **PROOF QUESTION.** Let V and W be vector spaces such that $\dim V = 4$ and $\dim W = 5$. Let U be a subspace of W such that $\dim U = 2$. Prove that there exists a linear function $T : V \to W$ such that $\operatorname{im} T = U$.