Math 129B, problem set 08
Outline due: Wed Apr 11
Due: Mon Apr 16
Last revision due: Mon May 14

Problems to be done, but not turned in: (6.4) 2, 6; (6.5) 1, 5, 6; (8.1) 2, 9, 11, 15.

Problems to be turned in:

1. Let $B = \left\{1, 2+x, 3-x+x^2\right\}$ and $B' = \left\{1+2x-x^2, x-3x^2, x^2\right\}$ be bases for $\mathbb{P}_2$ (i.e., you may take this as given). Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear function such that

$$[T]_{B,B} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix}.$$

   (a) Find a basis for $\operatorname{im} T$. Prove your answer.

   (b) Calculate $[T]_{B',B'}$. You can either use the change-of-basis formula or calculate $[T]_{B',B'}$ directly from its definition.

2. For a fixed positive integer n, let $T_n : \mathbb{P}_n \to \mathbb{P}_n$ be defined by the formula

$$T_n(p(x)) = p(x-1),$$

   and let $B = \{1, \dots, x^n\}$ be the standard basis for $\mathbb{P}_n$.

   (a) Describe $[T_n]_{B,B}$, the matrix of T_n relative to the basis B. (Justify your answer.)

   (b) Find an explicit formula for T_n^{-1}, and use that formula to describe $[T_n^{-1}]_{B,B}$, the matrix of T_n^{-1} relative to the basis B. (Justify your answers.)

   (c) Interpret the previous parts of this problem purely in terms of matrix multiplication. (Justify your answer.)

(cont. on other side)

3. Let $A = \begin{bmatrix} 14 & -18 \\ 9 & -\frac{23}{2} \end{bmatrix}$, and let $T = \mu_A$ be the usual linear function associated with A. Note that $B = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$. (This follows because neither vector is a scalar multiple of the other, making B linearly independent, and therefore a basis for $\mathbb{R}^2$, by the Two out of Three Theorem.)

   (a) Find $A' = [T]_{B,B}$, the matrix of T relative to the bases B and B.

   (b) Express $A' = [T]_{B,B}$ in terms of the original matrix A.

   (c) What happens to $(A')^n$ as $n \to \infty$?

   (d) Find a vector $\mathbf{v} \in \mathbb{R}^2$ such that, for any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, $A^n \begin{bmatrix} x \\ y \end{bmatrix}$ becomes arbitrarily close to a scalar multiple of $\mathbf{v}$ as $n \to \infty$.

4. Let $A = \begin{bmatrix} -8 & 20 & -10 \\ -5 & 12 & -5 \\ -5 & 10 & -3 \end{bmatrix}$. A calculation shows that the characteristic polynomial of A is $(\lambda - 2)^2(\lambda + 3)$ (you may assume this). Find the eigenvalues of A, and for each eigenvalue of A, find a basis for the associated eigenspace.

5. Let $V = \mathbb{D}^\infty(\mathbb{R})$, the space of infinitely differentiable functions on $\mathbb{R}$, and let $\Delta : V \to V$ be the linear function defined by $\Delta(f) = f''$. Prove that every $\lambda \in \mathbb{R}$ is an eigenvalue of Δ.

   Remark: In this context, Δ is often called the *Laplacian* on $\mathbb{R}$. The eigenvalues and eigenvectors (or in this context, *eigenfunctions*) of Δ play an important role in solving the equations governing heat, light, and sound, among other things.

6. Suppose that A is an $n \times n$ matrix such that $A^4 = I_n$. What are the possible eigenvalues of A? Prove your answer.