Math 129B, problem set 04
Outline due: Wed Feb 29
Due: Mon Mar 05
Last revision due: Wed Apr 18

Problems to be done, but not turned in: (3.4) 2, 4, 6, 9, 12; (3.5) 7, 10, 12, 17, 19, 21.

Problems to be turned in:

1. Let $f, g, h \in \mathbb{F}(\mathbb{R})$ be defined by

$$f(x) = |x| \qquad g(x) = \begin{cases} -\frac{1}{2}x & \text{if } x \leq 0, \\ -3x & \text{if } x > 0. \end{cases} \qquad h(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 2x+1 & \text{if } x > 0. \end{cases}$$

   Is $\{f, g, h\}$ linearly independent? Prove or disprove.

2. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, and let

$$S = \{ X \in \mathbb{M}(2,2) | \, AX = XA \}.$$

   It can be shown that S is a subspace of $\mathbb{M}(2,2)$ (i.e., you may assume this without proof). Find the dimension of S, with proof. (This will involve some computation.)

3. Let V be a vector space, and suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for V.
   (a) Is it true that every subspace W of V has a basis that consists of some subset of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$? Prove or give a counterexample.
   (b) Suppose we have a subspace W of V and a basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ for W. Is it true that there exists a basis B for V that contains $\{\mathbf{w}_1, \mathbf{w}_2\}$ (i.e., such that two of the vectors of B are $\mathbf{w}_1, \mathbf{w}_2$)? Prove or give a counterexample.

4. Let V be a vector space. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for V and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ is a set of vectors in V such that

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{w}_1 - \mathbf{w}_2, \\ \mathbf{v}_2 &= -5\mathbf{w}_1 + \mathbf{w}_2 + 2\mathbf{w}_3, \\ \mathbf{v}_3 &= 2\mathbf{w}_2 - 8\mathbf{w}_5, \\ \mathbf{v}_4 &= \mathbf{w}_2 - 3\mathbf{w}_3 + \mathbf{w}_5. \end{aligned}$$

   (a) Is it possible that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ is linearly independent? Prove or disprove.
   (b) Must it be the case that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ spans V? Prove or disprove.
   (c) Can you find a linearly independent subset of $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ with the same span as $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$? Prove or disprove.

5. (3.5) 13. (Thm. 3.15 is also known as the Two Out of Three Theorem.)

6. Let V be a vector space.

   (a) Prove that for $\{\mathbf{u}_1, \ldots, \mathbf{u}_s, \mathbf{y}\} \subseteq V$, if $\mathbf{y} \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_s\}$, then

   $$\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_s, \mathbf{y}\} \subseteq \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_s\}.$$

   (b) Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{x}, \mathbf{w}_1, \ldots, \mathbf{w}_r\} \subseteq V$. Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{x}\}$ is linearly independent and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_r\}$ spans V, then we can find some i such that $1 \leq i \leq r$ and

   $$\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_{i-1}, \mathbf{x}, \mathbf{w}_{i+1}, \ldots, \mathbf{w}_r\}$$

   spans V.