Sample Exam 2
Math 128B, Spring 2021

1. (12 points) Explain why each of the following polynomials in $\mathbf{Q}[x]$ is irreducible over $\mathbf{Q}$. Be specific: If you refer to a prime p, say which p you are using; if you use the fact that another polynomial is irreducible, explain why that other polynomial is irreducible.

(a) $x^3 - 17x + 13$

(b) $x^{13} + 55x^8 - 25x^3 + 30$

(c) $\dfrac{x^{10}}{21} + 5x^7 + 2x^3 + 3$

For questions 2–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (12 points) Let $f(x)$ be a polynomial in $\mathbf{Z}[x]$ with $\deg f(x) \geq 1$. If $f(x)$ is irreducible over $\mathbf{Z}$, then $f(x)$ is irreducible over $\mathbf{Q}$.

3. (12 points) Let F be a field, let V be a vector space over F, and let $\{v_1, v_2, v_3\}$ be a subset of V that is linearly independent and spans V. Then it is possible that V contains a linearly independent subset $\{w_1, w_2, w_3, w_4, w_5\}$.

4. (12 points) Let D be an integral domain, and let $a \in D$ be irreducible. Then it must be the case that a is prime.

5. (12 points) Let D be a unique factorization domain. Then it must be the case that every ideal of D has the form $\langle a \rangle$ for some $a \in D$.

6. (14 points) **PROOF QUESTION.** Let $f(x), g(x) \in \mathbf{Q}[x]$ be polynomials such that

$$\begin{array}{lllll} f(1) = 13, & f(-3) = 4, & f(7) = 22, & f(5) = -10, & \deg f = 3, \\ g(1) = 13, & g(-3) = 4, & g(7) = 22, & g(5) = -10, & \deg g = 3. \end{array}$$

Prove that $f(x) = g(x)$.

7. (14 points) **PROOF QUESTION.** In this question, you may take it as given that if I and J are ideals of a ring R, then $I \cap J$ is an ideal of R (i.e., you do not need to prove this).

Let D be a principal ideal domain, and let a and b be nonzero elements of D. Use $\langle a \rangle \cap \langle b \rangle$ to prove that there exists some $m \in D$ such that:

- a divides m and b divides m; and
- For $M \in D$, if a divides M and b divides M, then m divides M.