Sample Exam 1
Math 128B, Spring 2021

1. (12 points) Let I be the ideal $\langle x^3 + x + 1 \rangle$ in $\mathbf{Z}_2[x]$, and consider the quotient ring $\mathbf{Z}_2[x]/I = \mathbf{Z}_2[x]/\langle x^3 + x + 1 \rangle$. Find a polynomial $ax^2 + bx + c \in \mathbf{Z}_2[x]$ such that

$$(x^2 + 1 + I)(x + 1 + I) = ax^2 + bx + c + I$$

and each of a, b, c is either 0 or 1. Show all your work.

For questions 2–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (12 points) Let R be a commutative ring with unity, and let A be an ideal of R. If A is prime, then A must be maximal.

3. (12 points) Let R be a ring. Then it must be the case that for all $a, b \in R$, $ab = ba$.

4. (12 points) Let R and S be rings, let $\varphi : R \to S$ be a **surjective** ring homomorphism, and let $A = \{r \in R \mid \varphi(r) = 0\}$. Then A is an ideal of R, and R/A is isomorphic to S.

5. (12 points) If R is a commutative ring with unity, $a \in R$, and $a(a-1) = 0$, then it must be the case that either $a = 0$ or $a = 1$.

6. (14 points) **PROOF QUESTION.** Let

$$\mathbf{Q}[\sqrt{5}] = \left\{ a + b\sqrt{5} \mid a, b \in \mathbf{Q} \right\},$$

which is a subring of $\mathbf{R}$ (i.e., you may take this as given; do not prove it). Consider the function $\varphi : \mathbf{Q}[\sqrt{5}] \to \mathbf{Q}[\sqrt{5}]$ defined by

$$\varphi(a + b\sqrt{5}) = a - b\sqrt{5}$$

for all $a + b\sqrt{5} \in \mathbf{Q}[\sqrt{5}]$.

Prove that φ is a ring homomorphism.

7. (14 points) **PROOF QUESTION.** Let R be a commutative ring with **characteristic 2**, and let

$$I = \left\{ x \in R \mid x^2 = 0 \right\}.$$

Prove that I is an ideal of R.