

Math 128B, problem set 11
Outline due: Fri May 14
Due: Mon May 17
Last revision due: TBA

Problems to be done, but not turned in: (Ch. 32) 1–33 odd.

Problems to be turned in:

1. Let F be a field of characteristic 0, and suppose that E is the splitting field of some $f(x) \in F[x]$. Suppose the roots of $f(x)$ are $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in E$, and that $\text{Gal}(E/F)$ permutes the roots of f as

$$\text{Gal}(E/F) = \{e, (\alpha_1 \ \alpha_2), (\alpha_3 \ \alpha_4), (\alpha_1 \ \alpha_2)(\alpha_3 \ \alpha_4)\}.$$

Draw as much of the lattice of subfields of E containing F as you can derive from the given information. Also, for each subfield K_i , indicate $[K_i:F]$ and $\text{Gal}(E/K_i)$.

2. Let E be the splitting field of $x^4 + 1$ over \mathbf{Q} .
 - (a) Find $\text{Gal}(E/\mathbf{Q})$, and list all of its elements explicitly as permutations of roots of $x^4 + 1$.
 - (b) Find all subfields of E , and draw the lattice of subfields of E containing \mathbf{Q} . For each subfield K , write K in the form $K = \mathbf{Q}(a)$, indicate $[K:\mathbf{Q}]$ and find $\text{Gal}(E/K)$.
3. Let $E = \mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.
 - (a) Find $G = \text{Gal}(E/\mathbf{Q})$, and write them as permutations of $\pm\sqrt{d}$, where $d = 2, 3, 5$. No proof necessary.
 - (b) Choose a subgroup $H \leq G$ such that $|H| = 4$, and find E_H , the fixed field of H .
 - (c) Find $\text{Gal}(E/\mathbf{Q}(\sqrt{15}))$.
4. Let F be a field of characteristic 0, and let E be the splitting field of some $f(x) \in F[x]$ such that $\text{Gal}(E/F) \approx S_4$. Prove that E has a subfield K containing F such that $[K:F] = 4$.
5. Let E be the splitting field of some $f(x) \in \mathbf{Q}[x]$ such that $\text{Gal}(E/\mathbf{Q}) \approx A_4$.
 - (a) Draw the lattice of subfields of E , in the following sense: Let K_1, K_2, \dots be the subfields of E . Draw as much of the subfield lattice of E as you can derive from the given information. Also, for each subfield K_i , indicate $[K_i:\mathbf{Q}]$ and find $\text{Gal}(E/K_i)$.
 - (b) Which subfields of E are splitting fields of some $g(x) \in \mathbf{Q}[x]$? Prove your answer.
6. Let F be a field of characteristic 0, and let E be the splitting field of some $f(x) \in F[x]$ such that $\text{Gal}(E/F) \approx D_5$. Prove that E has a subfield K containing F with $K \neq E, F$ such that K is the splitting field of some $f(x) \in F[x]$. What is $[K:F]$?