Math 128B, problem set 10
Outline due: Fri May 07
Due: Mon May 10
Last revision due: TBA

Problems to be turned in:

1. Let $\alpha = (1\ 10\ 9\ 2)(3\ 5\ 6)(4\ 8)$ and $\beta = (1\ 10\ 7\ 6\ 5)(2\ 3\ 4\ 9\ 8)$ be elements of S_{10}.
   (a) Compute $\alpha\beta$ and α^{-1}, in cycle form.
   (b) Find the orders of α, β, and $\alpha\beta$.

2. For a group G and $\{a_1, \dots, a_n\} \subseteq G$, $\langle a_1, \dots, a_n \rangle$, called the *subgroup generated by* $\{a_1, \dots, a_n\}$, is the smallest subgroup of G containing $\{a_1, \dots, a_n\}$ (i.e., the intersection of all such subgroups). Put another way, $\langle a_1, \dots, a_n \rangle$ is the subgroup obtained by taking products and inverses of elements until nothing new is obtained.
   (a) Let $G_1 = \langle (1\ 2), (3\ 4) \rangle$ (i.e., the subgroup of S_4 generated by $(1\ 2)$ and $(3\ 4)$). Write out all elements of G_1, find all subgroups of G_1, and draw the subgroup lattice of G_1.
   (b) Let $G_2 = \langle (1\ 2), (3\ 4), (5\ 6) \rangle$ (i.e., the subgroup of S_6 generated by $(1\ 2)$, $(3\ 4)$, and $(5\ 6)$). Write out all elements of G_2. Find all **cyclic** subgroups of G_2.
   (c) Give an example of a subgroup of G_2 of order 4.

3. (a) Prove that all proper subgroups of S_3 are cyclic.
   (b) Find all subgroups of S_3 and draw its subgroup lattice.

4. Let $D_4 = \{e, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2), (2\ 4), (1\ 3), (1\ 2)(3\ 4), (1\ 4)(2\ 3)\}$ be the symmetries of a square, represented as permutations of its vertices.
   (a) Find all cyclic subgroups of D_4.
   (b) Recall that for any rotation r and reflection f, $frf^{-1} = r^{-1}$. With that in mind, what is $Z(D_4)$?
   (c) Draw the subgroup lattice of D_4. Make sure to include the subgroup

$$V = \{e, (1\ 3)(2\ 4), (1\ 2)(3\ 4), (1\ 4)(2\ 3)\}.$$

5. Prove that A_4 has exactly one nontrivial proper normal subgroup, and identify exactly what that subgroup is.

6. (a) List the conjugacy classes of S_5, and for each class, determine how many elements of S_5 are in that conjugacy class.
   (b) Let N be a normal subgroup of S_5, and let $H = N \cap A_5$. Prove that H is a normal subgroup of A_5 and that either $N = H \leq A_5$ or $|N : H| = 2$.
   (c) Prove that the only normal subgroups of S_5 are $\{e\}$, A_5, and S_5.