Math 128B, problem set 08
Outline due: Thu Apr 15
REVISED FRI APR 16
Due: Mon Apr 18
Last revision due: Mon May 17

Problems to be done, but not turned in: (Ch. 21) 1–49 odd.

1. Prove that if a is algebraic over F, $b \in F$, and $b \neq 0$, then $\dfrac{a}{b}$ is algebraic over F.
2. Let E be the splitting field of $x^3 - 11$ over $\mathbf{Q}$. Find three distinct proper subfields of E, and draw the corresponding tower of subfields, with degrees, as in Example 4 of Ch. 21. (No proof necessary.)
3. Let $\alpha = \sqrt{5} - \sqrt{2}$. Find the minimal polynomial of α over $\mathbf{Q}$, find $[\mathbf{Q}(\alpha) : \mathbf{Q}]$, and find a basis for $\mathbf{Q}(\alpha)$ over $\mathbf{Q}(\sqrt{10})$ (all with proof/justification).
4. Suppose F is a field, a is an element of some extension of F, and $[F(a) : F] = 7$.
   (a) Prove that $F(a^3) = F(a)$.
   (b) Is it necessarily true that $F(a^4) = F(a)$? Generalize.
5. Let $a = \sqrt{2} + \sqrt{3}$ and $b = \sqrt{3} + \sqrt{5}$. (See also Ch. 21, problem 15.)
   (a) Find at least 9 distinct proper subfields of $E = \mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$, including:
      - At least 4 subfields K of E such that $[K : \mathbf{Q}] = 4$; and
      - At least 4 subfields K of E such that $[K : \mathbf{Q}] = 2$;

      and draw the corresponding tower of subfields, with degrees, as in Example 4 of Ch. 21.
   (b) Indicate where $\mathbf{Q}(a)$ and $\mathbf{Q}(b)$ fit into the tower of fields from part (a).
   (c) Prove that $[\mathbf{Q}(a, b) : \mathbf{Q}] < [\mathbf{Q}(a) : \mathbf{Q}][\mathbf{Q}(b) : \mathbf{Q}]$.
6. (Ch. 21) 24.
7. (Ch. 21) 34.