Math 128B, problem set 07
Outline due: Fri Apr 09
Due: Mon Apr 12
Last revision due: Mon May 10

Problems to be done, but not turned in: (Ch. 20) 1–43 odd.

Problems to be turned in:

1. Find the splitting field of $f(x) = x^4 - 4x^2 - 5$ over $\mathbf{Q}$. Prove your answer, and state your answer in the form $\mathbf{Q}(a, b)$ or $\mathbf{Q}(a)$.

2. Find the splitting field of $x^6 + 1$ over $\mathbf{Q}$. Prove your answer, and state your answer in the form $\mathbf{Q}(a, b)$ or $\mathbf{Q}(a)$. You may find it helpful to use complex numbers of the form $\omega = e^{2\pi i/n}$; note that for such a complex number, $\omega^n = 1$.

3. Let α be a root of $f(x) = x^3 + x^2 + 1$ in the field $\mathbf{Z}_2[x]/\langle f(x)\rangle$. Write out a multiplication table for $\mathbf{Z}_2(\alpha)$. (In particular, express all of the elements of $\mathbf{Z}_2(\alpha)$ in terms of α.)

4. (Ch. 20) 12. (Do 11 first.)

5. Let E be an extension field of $\mathbf{Z}_2$.

   (a) Prove that if $f(x) \in \mathbf{Z}_2[x]$, $\beta \in E$, and $f(\beta) = 0$, then $f(\beta^2) = 0$. (Suggestion: Use the field automorphism $\rho : E \to E$ given by $\rho(x) = x^2$.)

   (b) Now suppose α is a root of $f(x) = x^4 + x^3 + 1$ in E. Factor $f(x)$ into linear factors over E.

6. Find a basis for $\mathbf{Q}(\sqrt[6]{2})$ as an extension of $\mathbf{Q}(\sqrt{2})$. Prove your answer.

7. (Ch. 20) 36.