Math 128B, problem set 06
Outline due: Wed Mar 17
Due: Mon Mar 22
Last revision due: Mon May 10

Problems to be done, but not turned in: (Ch. 19) 1–37 odd.

Problems to be turned in:

1. (Ch. 18) 8.

2. (Ch. 18) 22.

3. Let D be a principal ideal domain, and let a, b be nonzero elements of D. Prove that there exists some $g \in D$ (depending on a and b) such that:

   - The element g divides both a and b;
   - For $d \in D$, if d divides both a and b, then d divides g; and
   - There exist $x, y \in D$ such that $ax + by = g$.

   (Note: Because of the first two properties, we may think of g as $\gcd(a, b)$, the greatest common divisor of a and b.)

   Suggestion: Consider $\langle a \rangle + \langle b \rangle$, which we know is an ideal of D (Ch. 14 #10).

4. Let F be a field, and suppose that for $p(x), q(x) \in F[x]$, we have that $\gcd(p(x), q(x)) = 1$ (see problem 3). Prove that for any $f(x) \in F[x]$, there exist $r(x), s(x) \in F[x]$ such that
$$\frac{f(x)}{p(x)q(x)} = \frac{r(x)}{p(x)} + \frac{s(x)}{q(x)}.$$
   (Note: Suitably refined, this is the theorem that makes the integration technique of partial fractions possible.)

5. Let W be the subspace of $\mathbf{F}_2^9$ defined by
$$W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

   (a) How many vectors are there in W? Explain your answer.

   (b) What is the dimension d of W? Explain your answer.

(continued on next page)

6. Let V be a vector space over the field F, and let $\{v_1, \ldots, v_k\} \subseteq V$ be linearly independent.

   (a) Prove that the following are equivalent:

   - $x \in V$ is a linear combination of $v_1, \ldots, v_k$.
   - $\{v_1, \ldots, v_k, x\}$ is linearly dependent.

   (b) To say that $\{v_1, \ldots, v_k\}$ is a *maximal linearly independent set* means that $\{v_1, \ldots, v_k\}$ is linearly independent, and for any $x \in V$, $\{v_1, \ldots, v_k, x\}$ is linearly dependent. Prove that a maximal linearly independent set in V is a basis for V.

7. Suppose $\{v_1, \ldots, v_\ell\} \subseteq V$ is linearly independent and $\{w_1, \ldots, w_s\}$ spans V.

   (a) Prove (by contradiction) that if $k < \ell$, then $\{v_1, \ldots, v_k\}$ cannot span V.

   (b) Suppose, for some $k < \ell$, we know that $\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_s\}$ spans V. (Note that by part (a), we must have $s > k$.) Prove that, possibly after renumbering $w_{k+1}, \ldots, w_s$, we have that $\{v_1, \ldots, v_k, v_{k+1}, w_{k+2}, \ldots, w_s\}$ also spans V.

   (c) Prove by induction on k that for $0 \leq k \leq \ell$, we can renumber $w_1, \ldots, w_s$ so that

   $$\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_s\}$$

   spans V. Note that the base case $k = 0$ is vacuously true, so you just need to prove the induction step.

   Remark: As a consequence of this problem, $\ell \leq s$.