Math 128B, problem set 05
Outline due: Wed Mar 10
Due: Mon Mar 15
Last revision due: Mon May 10

Problems to be done, but not turned in: (Ch. 17) 21–43 odd. (Ch. 18) 1–47 odd.

Problems to be turned in:

1. (a) For $f(x) \in \mathbf{Z}_p[x]$ and $\deg f = 2$ or 3, explain how to check if $f(x)$ is irreducible without doing any polynomial long division.
   (b) Use part (a) to find all irreducible polynomials of degree 2 and 3 in $\mathbf{Z}_2[x]$.
   (c) Fill in the blanks and prove: For $f(x) \in \mathbf{Z}_2[x]$, if $\deg f$ is at most _______, then we can check if $f(x)$ is irreducible using at most one long division, and if $\deg f$ is at most _______, then we can check if $f(x)$ is irreducible using at most three long divisions.
2. Construct a field of order 125. Carefully justify your claims.
3. (Ch. 17) 14.
4. (**SUGGESTIONS ADDED MAR 15**) Let $f(x) = x^6 - 6x + 4$. Note that $f(x)$ does *not* satisfy Eisenstein's Criterion.
   (a) Prove that if $f(x) = g(x)h(x)$, for $g, h \in \mathbf{Z}[x]$, then every coefficient of both g and h must be even except for the leading coefficient. (Suggestion: Try imitating Example 4 on p. 315.)
   (b) Prove that f is irreducible over $\mathbf{Q}$. (Suggestion: Try using the Rational Root Theorem (Ch. 17, prob. 31) and reducing mod 4.)
5. Find (with proof) an example of a quintic $x^5 + ax + b$, $a, b \in \mathbf{Z}$ that is irreducible over $\mathbf{Q}$ and has precisely three real zeros. Use everything you've ever learned, especially calculus. Try to make your example different from the examples of everyone else in the class.
6. Since $3\mathbf{Z}$ is a ring without unity, for $a \in 3\mathbf{Z}$, we say that a is *reducible* in $3\mathbf{Z}$ if $a = bc$ for some $b, c \in 3\mathbf{Z}$; otherwise, we say that a is *irreducible* in $3\mathbf{Z}$.
   (a) List the irreducible elements of $3\mathbf{Z}$. (Prove/explain your answer.)
   (b) Find some $n \in 3\mathbf{Z}$ such that $n = ab = cd$ and $a, b, c, d > 0$ are all distinct irreducible elements of $3\mathbf{Z}$.
7. (Ch. 18) 2.