Math 128B, problem set 04
Outline due: Wed Mar 03
Due: Mon Mar 08
Last revision due: Mon Apr 05

Problems to be done, but not turned in: (Ch. 16) 1–61 odd. (Ch. 17) 1–19 odd.

Problems to be turned in:

Throughout this problem set, you may assume the (as yet unproven) fact that if F is a field, then polynomials in $F[x]$ factor uniquely into irreducible polynomials.

1. Let F be a field, let R be a commutative ring with unity, and suppose that $\varphi : F \to R$ is a ring homomorphism with $\varphi(1) = 1$. Prove that φ is injective.

2. Let F be a field. Fill in the blank and prove: For any positive integer n, there are at most _______ elements of F that are equal to their nth powers.

3. Let F be a subfield of $\mathbf{C}$. (Note that $\mathbf{Z}$ must therefore be a subring of F.)

   (a) Find a cubic polynomial $f(x) \in F[x]$ such that $f(2) = 0$, $f(3) = 0$, $f(4) = 0$, and $f(5) = 17$. Your formula should express f as a product of linear polynomials.

   (b) Find a cubic polynomial $g(x) \in F[x]$ such that $g(2) = 0$, $g(3) = -31$, $g(4) = 0$, and $g(5) = 17$. Your formula should express g as the sum of two polynomials of the form used in part (a).

   (c) Let a_i $(1 \le i \le 4)$ be distinct elements of F, and let b_i $(1 \le i \le 4)$ be elements of F, not necessarily distinct. Prove that there exists a **unique** cubic polynomial $h(x)$ such that $h(a_i) = b_i$ for $1 \le i \le 4$.

4. (Ch. 16) 48.

5. (Ch. 17) 20.

6. Explain your answers in each part.

   (a) (Ch. 17) 24(a).

   (b) Determine the number of polynomials of the form $(x - a)(x^2 + bx + c)$ in $\mathbf{Z}_p[x]$, where $x^2 + bx + c$ is irreducible.

   (c) Determine the number of polynomials of the form $(x - a)(x - b)(x - c)$ in $\mathbf{Z}_p[x]$. (Watch out for repetitions!)

   (d) Determine the number of irreducible polynomials over $\mathbf{Z}_p$ of the form $x^3 + ax^2 + bx + c$.