Math 128B, problem set 02
UPDATED WED FEB 10
Outline due: Wed Feb 10
Due: Mon Feb 15
Last revision due: Mon Apr 05

Problems to be done, but not turned in: (Ch. 13) 29–63 odd; (Ch. 14) 1–71 odd.
Fun: (Ch. 13) 66. (Ch. 14) 70.

Problems to be turned in:

1. For p prime, let
$$\mathbf{Z}_p[i] = \{a + bi \mid a, b \in \mathbf{Z}_p\}$$
with $i^2 = -1$ be a ring with p^2 elements.
   (a) Write down the multiplication table for $\mathbf{Z}_2[i]$. Is $\mathbf{Z}_2[i]$ an integral domain? Is $\mathbf{Z}_2[i]$ a field?
   (b) Now considering arbitrary p again, find a formula for the multiplicative inverse of the element $a+bi$ of $\mathbf{Z}_p[i]$, and prove your formula works except under certain conditions. Under what conditions does this formula fail? (Suggestion: Imitate the standard method for finding inverses in the complex numbers.)
   (c) Complete the following: "$\mathbf{Z}_p[i]$ is a field if and only if the equation $x^2 = ?$ has no solutions in $\mathbf{Z}_p$." (Proof not necessary.)
2. (Ch. 13) 50 (relies on problems 15 and 49).
3. (Ch. 14) 12.
4. (Ch. 14) 48.
5. Suppose that I is a *proper* ideal of $\mathbf{R}[x]$, and that x^3+1 and x^2+4x+3 are elements of I. What are the possibilities for I? Justify your answer, and for each possibility, find a generating set that is as small as possible.
6. Let $R = \mathbf{Z}_2[x]/\left\langle x^3 + x + 1\right\rangle$.
   (a) Write down the elements of R. How many elements are there?
   (b) Prove that R is a field by explicitly finding an inverse for each element of R.
   (c) Find a cubic polynomial $f(x) \in \mathbf{Z}_2[x]$ such that $\mathbf{Z}_2[x]/\langle f(x)\rangle$ is not a field. Prove your answer.
7. (Ch. 14) 28.