This test consists of 7 questions on 2 pages, totalling 100 points. You are allowed to use one page of notes. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

1. (13 points) Consider the polynomial $f(x) = x^5 - 3$ in $\mathbf{Q}[x]$. Find a, b such that $f(x)$ splits in $\mathbf{Q}(a, b)$, and write $f(x)$ as a product of linear factors in $\mathbf{Q}(a, b)$. (You may find the abbreviation $\omega_n = e^{2\pi i/n}$ helpful.)

For questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (13 points) **TRUE/FALSE:** The multiplicative group of the field of order $64 = 2^6$ contains an element of (multiplicative) order 4.

3. (13 points) **TRUE/FALSE:** Let F be a field, let E be an extension of F, and let $a, b \in E$. If the minimal polynomial of b over F has degree 7, and $[F(a):F] = 5$, then it must be the case that $[F(a,b):F(a)] = 7$.

4. (13 points) **TRUE/FALSE:** Let $\alpha = \sqrt[4]{7}$, and let E be an extension of $\mathbf{Q}$ such that $\mathbf{Q} \subseteq E \subseteq \mathbf{Q}(\alpha)$ and $\alpha \notin E$. Then it must be the case that $\{1, \alpha, \alpha^2, \alpha^3\}$ is a basis for $\mathbf{Q}(\alpha)$ as a vector space over E.

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5. (16 points) **PROOF QUESTION.** Let α be a complex number such that $\alpha^5 - 7\alpha^3 + 14 = 0$.

(a) Find a basis for $\mathbf{Q}(\alpha)$ as a vector space over $\mathbf{Q}$, and **prove** your answer.

(b) Prove that $\alpha^4 - 5\alpha^2 + \alpha - 13 \neq 0$.

6. (16 points) **PROOF QUESTION.** Let $GF(243)$ be the field of order $243 = 3^5$.

(a) List all subfields of $GF(243)$. No explanation necessary.

(b) Let α be a primitive element of $GF(243)^*$, and let $\beta = \alpha^{11}$. What is the order of β? Justify your answer.

(c) Prove that $\mathbf{Z}_3(\beta) = GF(243)$.

7. (16 points) **PROOF QUESTION.** Let F be a field and let E be an extension of F such that $[E : F] = 9$. Let a be an element of E such that for any $f(x) \in F[x]$ with $\deg f(x) \leq 3$, $f(a) \neq 0$. Prove that $E = F(a)$.