Math 128B, Spring 2021 **Exam 2**

This test consists of 7 questions on 2 pages, totalling 100 points. You are allowed to use one page of notes. Unless otherwise stated, you may take as given anything that has been proven in class, in the homework, or in the reading.

1. (13 points) In this problem, be specific: If you refer to a prime p, say which p you are using; if you use the fact that another polynomial is irreducible, explain why that other polynomial is irreducible.

(a) Find an integer c such that $f(x) = x^5 - 35x^4 + c$ is irreducible over $\mathbf{Q}$. Briefly **JUSTIFY** your answer.

(b) Let d be an odd integer. Briefly **EXPLAIN** how you can be sure that $x^3 - x^2 + d$ is irreducible over $\mathbf{Q}$.

No explanation necessary.

For questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (13 points) **TRUE/FALSE:** Let D be an integral domain and $a, b, c, d \in D$. If d is irreducible, d divides a, and $a = bc$, then it must be the case that either d divides b or d divides c.

3. (13 points) **TRUE/FALSE:** Let V be a vector space over a field F. It is possible that V contains vectors v_1, v_2, v_3, v_4 and w_1, w_2, w_3 such that $\{v_1, v_2, v_3, v_4\}$ is a basis for V and $\{w_1, w_2, w_3\}$ spans V.

4. (13 points) **TRUE/FALSE:** Let $p(x) \in \mathbf{Z}_7[x]$ be a polynomial of degree 3 such that

$$p(a) \neq 0 \pmod 7$$

for $0 \leq a \leq 6$. Then it must be the case that $\mathbf{Z}_7[x]/\langle p(x) \rangle$ is a field.

---

5. (16 points) **PROOF QUESTION.** Let D be an integral domain and $a, b, c \in D$. Suppose that $a = bc$.

(a) Prove that $\langle a \rangle \subseteq \langle b \rangle$.

(b) Now suppose we additionally assume that $\langle b \rangle \subseteq \langle a \rangle$. Prove that in that case, c is a unit.

6. (16 points) **PROOF QUESTION.** Let F be a field. Fill in the blank and prove: Let k be a positive integer. There are at most _____ elements $a \in F$ such that $a^k = 1$.

7. (16 points) **PROOF QUESTION.** Let

$$I = \{ f(x) \in \mathbf{Q}[x] \mid f(2) = 0 \text{ and } f(5) = 0 \}.$$

Find some $g(x) \in \mathbf{Q}[x]$ such that $I = \langle g(x) \rangle$, and prove $I = \langle g(x) \rangle$.