Sample Final Exam
Math 128A, Fall 2020

Note that this exam began with a "recite the definition" problem, which is a type of problem that your exam will not have. On the other hand, there will most probably be some kind of computational problem where it will be helpful to know definitions well, so either memorizing or carefully writing down definitions in your notes will be helpful.

1. (12 points) Let R be a ring and let A be an ideal of R. Define the factor ring (quotient ring) R/A. In particular, define the elements of R/A and the operations of R/A.

2. (15 points) Let

$$\alpha = (1\ 10\ 8)(2\ 4)(3\ 11)(5\ 12\ 7\ 6\ 9)$$
$$\beta = (2\ 7\ 3\ 9\ 6\ 10\ 12\ 4)(5\ 11\ 8)$$

Calculate $\alpha\beta$, α^{-1}, and $|\alpha|$. Put all permutation answers in cycle form, and show all your work.

3. (15 points) List all abelian groups of order $400 = 2^4 5^2$, up to isomorphism. Show all your work.

For questions 4–9, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

4. (13 points) **(TRUE/FALSE)** If a and b are elements of the dihedral group D_n for some $n \geq 3$, then it must be the case that $ab = ba$.

5. (13 points) **(TRUE/FALSE)** Suppose G and $\overline{G}$ are groups and $\varphi : G \to \overline{G}$ is a surjective (onto) homomorphism. If G is abelian, then it must be the case that $\overline{G}$ is abelian.

6. (13 points) **(TRUE/FALSE)** It is possible that there exists a cyclic group G such that G is isomorphic to $\mathbf{Z}_3 \oplus \mathbf{Z}_{12}$.

7. (13 points) **(TRUE/FALSE)** Let R be a ring with multiplicative identity 1. If x is an element of R such that $x^2 = x$, then it must be the case that either $x = 0$ or $x = 1$.

8. (13 points) **(TRUE/FALSE)** Let G be a group and let a be an element of G of finite order n. If k divides n, then it must be the case that the order of a^k is $\dfrac{n}{k}$.

9. (13 points) **(TRUE/FALSE)** Let G be a finite group and let H be a subgroup of G. If a and b are elements of G such that $aH = bH$, then it must be the case that $a = b$.

10. (16 points) **PROOF QUESTION.** Recall that $Z(G)$, the center of G, is the subgroup of all $a \in G$ such that $ax = xa$ for all $x \in G$.

Suppose G is a group, N is a normal subgroup of G, and a is an element of $Z(G)$ (the center of G). Prove that aN is an element of $Z(G/N)$ (the center of the factor group G/N).

11. (16 points) **PROOF QUESTION.** Suppose that G is a group, H is a subgroup of G, and a is a fixed element of G such that $ax = xa$ for all $x \in G$. Define

$$L = \{ha^n \mid h \in H,\ n \in \mathbf{Z}\}.$$

(In other words, L is the set of all products of an element of H and a power of a.)

Prove that L is a subgroup of G.

12. (16 points) **PROOF QUESTION.** Let G and $\overline{G}$ be groups, and let $\varphi : G \to \overline{G}$ be a homomorphism. Suppose H is a subgroup of G such that $|H| = 10$ and $a \in H$. Prove that $\varphi(a)^{10} = \overline{e}$, where $\overline{e}$ is the identity element of $\overline{G}$.

13. (16 points) **PROOF QUESTION.** Let G be a group of order 35. Prove that G contains an element of order 5.

14. (16 points) **PROOF QUESTION.** Recall that $\mathbf{F}(\mathbf{R})$ is the ring of all real-valued functions $f : \mathbf{R} \to \mathbf{R}$, with addition and multiplication defined pointwise (i.e., $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$). Recall also that $\mathbf{F}(\mathbf{R})$ is a ring (i.e., you may take this as given).

Let

$$S = \{f \in \mathbf{F}(\mathbf{R}) \mid f(3) = 0\},$$

or in other words, let f be the set of all $f : \mathbf{R} \to \mathbf{R}$ such that $f(3) = 0$. Prove that S is a subring of $\mathbf{F}(\mathbf{R})$.