Sample Exam 3
Math 128A, Fall 2020

Note that this exam began with a "recite the definition" problem, which is a type of problem that your exam will not have. On the other hand, there will most probably be some kind of computational problem where it will be helpful to know definitions well, so either memorizing or carefully writing down definitions in your notes will be helpful.

1. (12 points) Let G be a group of permutations of a set S. For $i \in S$:

(a) Define $\mathrm{orb}_G(i)$, the orbit of i under G.

(b) Define $\mathrm{stab}_G(i)$, the stabilizer of i in G.

(c) State the Orbit-Stabilizer Theorem.

2. (10 points) Let $G = \mathbf{Z}_3 \oplus \mathbf{Z}_9$.

(a) Find the number of elements of order 3 in G.

(b) Find the number of cyclic subgroups of order 3 in G.

No explanation necessary, but show all your work.

For questions 3–5, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (12 points) **TRUE/FALSE:** Let H be a subgroup of D_6. Then it must be the case that H is normal in D_6.

4. (12 points) **TRUE/FALSE:** Let $\varphi : \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to \mathbf{Z}_4 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_4$ be defined by

$$\varphi((0,0)) = (0,0,0), \qquad \varphi((1,0)) = (2,0,2),$$
$$\varphi((0,1)) = (2,2,0), \qquad \varphi((1,1)) = (0,0,2).$$

Then φ is a homomorphism.

5. (12 points) **TRUE/FALSE:** The group $\mathbf{Z}_{30} \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_{36}$ is isomorphic to the group $\mathbf{Z}_{45} \oplus \mathbf{Z}_{12} \oplus \mathbf{Z}_6$.

6. (14 points) **PROOF QUESTION.** Suppose G and $\overline{G}$ are groups, $\varphi : G \to \overline{G}$ is a homomorphism with $|\ker(\varphi)| = 2$, and a is an element of G such that the order of $\varphi(a)$ is 7. Prove that the order of a is either 7 or 14.

7. (14 points) **PROOF QUESTION.** Let $G = \mathbf{Z}_2 \oplus \mathbf{Z}_8$, and let $N = \langle (1,4) \rangle$.

(a) What is the order of the factor group G/N?

(b) Carefully prove that G/N is cyclic.

8. (14 points) **PROOF QUESTION.** Let G be a group of order 65. Prove that G must have an element of order 5.