Math 128A, problem set 09
Outline due: Mon Nov 09
Due: Mon Nov 16
Last revision due: Mon Dec 14

Problems to be done, but not turned in: (Ch. 10) 1–65 odd.
Fun: (Ch. 10) 42.

Problems to be turned in:

1. Note that if $r = R_{20}$ and f is any reflection in D_{18}, we know that $r^n = f^2 = e$ and $frf^{-1} = r^{-1}$, and that
$$D_{18} = \left\{r^i, r^i f \mid 0 \le i < 18\right\}.$$
(See PS08 for the case of D_{12} instead of D_{18}; D_{18} works the same.)
Let $H = \langle r^9 \rangle = \left\{e, r^9\right\}$, and let $K = \left\{r^{2i}, r^{2i} f \mid 0 \le i < 9\right\}$. You may assume that H and K are subgroups of D_{18}.
   (a) Prove that H and K are normal in D_{18}.
   (b) Prove that $D_{18} \approx H \oplus K$.

   (While you do not need to prove this, it is a fact that $H \approx Z_2$ and $K \approx D_9$, which means that $D_{18} \approx Z_2 \oplus D_9$.)
2. Suppose we define a function $\varphi : \mathbf{Z}_7 \to \mathbf{Z}_{20}$ by the formula $\varphi(x) = 2x$, i.e., $\varphi(0) = 0$, $\varphi(1) = 2$, $\varphi(2) = 4$, $\varphi(3) = 6$, $\varphi(4) = 8$, $\varphi(5) = 10$, $\varphi(6) = 12$. Is φ a homomorphism? Prove or disprove.
3. (Ch. 10) 16.
4. (Ch. 10) 24.
5. (Ch. 10) 30.
6. Suppose $\varphi : G \to \overline{G}$ is a homomorphism with $|\ker \varphi| = 105$, and suppose $a \in G$ has order 126. List all possibilities for the order of $\varphi(a)$, and prove your answer.
7. Suppose G is a group, $H \lhd G$, $K \lhd G$, $G = HK$, and $H \cap K = \{e\}$.
   (a) Prove that for $h \in H$ and $k \in K$, $hkh^{-1}k^{-1} \in H \cap K$.
   (b) Prove that if $h_1 k_1 = h_2 k_2$ for $h_1, h_2 \in H$ and $k_1, k_2 \in K$, then $h_1 = h_2$ and $k_1 = k_2$. (In other words, every element of G can be expressed in the form hk with $h \in H$ and $k \in K$ in exactly one way.)
   (c) Now define a map $\varphi : G \to H \oplus K$ by the formula
   $$\varphi(hk) = (h, k) \qquad \text{for } h \in H,\ k \in K.$$
   Prove that φ is a homomorphism.
   (d) Calculate $\ker \varphi$ (with proof). What can you conclude from the First Isomorphism Theorem?