Math 128A, problem set 07
Outline due: Mon Oct 26
Due: Wed Oct 28
Last revision due: Mon Nov 23

Problems to be done, but not turned in: (Ch. 7) 19–65 odd; (Ch. 8) 1–17 odd.
Fun: (Ch. 7) 38.

Problems to be turned in:

1. (Ch. 7) 40.
2. (Ch. 7) 44.
3. Prove that a group of order 20 must have an element of order 2.
4. Consider D_6 as a group of permutations of the vertices of a regular hexagon. For $1 \leq i \leq 6$, let
$$C_i = \{\alpha \in D_6 \mid \alpha(1) = i\}.$$
In other words, C_i is the set of all elements that send vertex 1 to vertex i.
For $1 \leq i \leq 6$, list all elements in C_i. What pattern(s) do you see? Do the sets C_i look familiar to you?
5. Let G be a group, and let H and K be subgroups of G. Recall that
$$HK = \{hk \mid h \in H, k \in K\}.$$
Recall also that HK is sometimes a subgroup of G, and sometimes not. This problem provides an alternative proof of the fact that $|HK| = \dfrac{|H|\,|K|}{|H \cap K|}$.
   (a) Prove that $HK = \bigcup_{h \in H} hK$, i.e, HK is the union of all cosets hK such that $h \in H$.
   (b) Let $S = \{hK \mid h \in H\}$. Prove that H can be considered as a group of permutations of S, or in other words, for any fixed $h \in H$, the map
   $$f_h(h'K) = hh'K$$
   is a bijection from S to itself.
   (c) What is $\mathrm{orb}_H(K)$? What is $\mathrm{stab}_H(K)$? (I.e., for which h is $hK = K$)? Explain.
   (d) Use the Orbit-Stabilizer Theorem to prove that $|HK| = \dfrac{|H|\,|K|}{|H \cap K|}$.
6. Is $\mathbf{Z}_5 \oplus \mathbf{Z}_5 \oplus \mathbf{Z}_{25}$ isomorphic to $\mathbf{Z}_{25} \oplus \mathbf{Z}_{25}$? Why or why not? Prove your answer.
7. Find the number of elements of order 6 and the number of cyclic subgroups of order 6 in $\mathbf{Z}_{15} \oplus \mathbf{Z}_{24}$.