Math 128a, problem set 06
Outline due: Fri Oct 09
Due: Wed Oct 14
Last revision due: Mon Nov 23

Problems to be done, but not turned in: (Ch. 6) 7–67 odd; (Ch. 7) 1–17 odd.
Fun: (Ch. 6) 64, 66.

Problems to be turned in:

1. Does there exist an automorphism $\varphi : \mathbf{Z}_{70} \to \mathbf{Z}_{70}$ such that $\varphi(17) = 21$? If so, describe *all* such φ as precisely as possible, with proof; if not, prove that no such φ exists.

2. Consider the groups $U(15)$, $U(20)$, and $U(24)$. For any two of them that you think are *not* isomorphic, prove that they are not isomorphic.

3. Find three groups G, H, K of order 24 such that $G \not\approx H$, $H \not\approx K$, and $G \not\approx K$. Prove your result.

4. Consider the group D_6, using our standard notation.

   (a) Let $K = \{e, F_{12}\} = \langle F_{12} \rangle$. List all of the left cosets of K and all of the right cosets of K.

   (b) Let $H = \{e, R_{120}, R_{240}\} = \langle R_{120} \rangle$. List all of the left cosets of H and all of the right cosets of H. Do you see any significant qualitative differences between this example and the previous one? Explain.

5. Let $H = 5\mathbf{Z} = \{n \in \mathbf{Z} \mid n = 5k \text{ for some } k \in \mathbf{Z}\}$. List all of the cosets of H in $\mathbf{Z}$. How many are there? Generalize as much as possible, both in terms of numbers of cosets and what those cosets are.

6. Let G be a group, and let H and K be subgroups of G such that $|H| = 60$ and $|K| = 70$. What are the possibilities for the order of $H \cap K$? Generalize.

7. (a) Let G be a group such that every nontrivial element of G has order 2. Prove that G is abelian.

   (b) Now let G be a group of order 8. Prove that if G is *not* abelian, then G must have an element of order 4.