Paragraph HW 10
Using Gauss' Law to calculate electric fields
Math 112, Spring 2006

Make sure you do SA25 before doing this paragraph homework.

Suppose we have a collection of stationary charges in $\mathbb{R}^3$. Coulomb's Law states that the electric field induced by a charge q at a position (x_0, y_0, z_0) on a point (x, y, z) is

$$\mathbf{E}(x,y,z) = \frac{q\mathbf{r}}{r^2},$$

where $\mathbf{r}$ is a unit vector pointing from (x_0, y_0, z_0) to (x, y, z), and r is the distance between (x_0, y_0, z_0) and (x, y, z). Here, $\mathbf{E}$ is given in the units of dynes (force) per esu (electrostatic unit), and q is in esus.

In principle, we can extend Coulomb's law to a continuous distribution of charges; specifically, if $\delta(x, y, z)$ is the density of charge at the point (x, y, z), in esus per cm^3, then

$$\mathbf{E}(a,b,c) = \iiint_W \frac{\delta(x,y,z)\mathbf{r}}{r^2}\,dx\,dy\,dz,$$

where W is a region enclosing all charges, and again, $\mathbf{r}$ is a unit vector pointing from (x, y, z) to (a, b, c), and r is the distance between (x, y, z) and (a, b, c), in cm. In practice, this integral (a triple integral of a vector function!) is too unwieldy to use for straightforward computation, and we will not use it here.

As we have seen already, the integral form of Gauss' Law states that for any closed (boundary-free) surface S,

$$\iint_S \mathbf{E}\cdot d\mathbf{S} = 4\pi q,$$

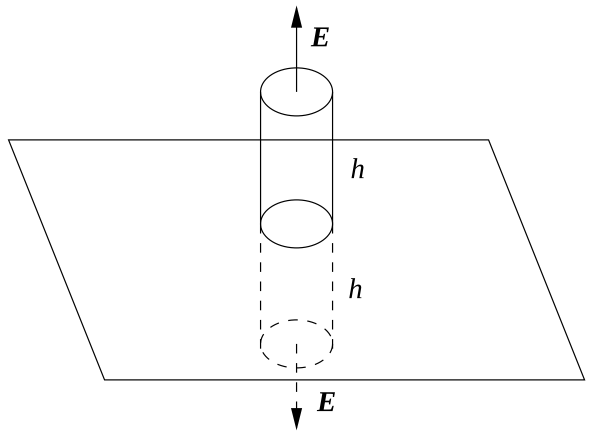
where q is the total amount of charge contained in S. This homework explores ways to compute $\mathbf{E}$ using Gauss' Law, symmetry, and the following fact about surface integrals: If S is a surface oriented by the unit normal $\mathbf{n}$, and $\mathbf{F}$ is a vector field such that $\mathbf{F}\cdot\mathbf{n}$ is constant at every point of S, then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = (\mathbf{F}\cdot\mathbf{n})(\text{surface area}(S)).$$

1. Suppose we have an infinite flat plate of charge; in other words, assume the entire xy-plane has a uniform charge density of δ esus per cm^2. By symmetry, the electric field $\mathbf{E}(x, y, z)$ $(z \neq 0)$ must be in the $\mathbf{k}$ direction, since at any point (x, y, z), the xy-plane looks the same in any horizontal direction. Note also that by symmetry, $\mathbf{E}(x, y, z)$ does not depend on x or y.

   Use Gauss' Law and the above fact about surface integrals to calculate $\mathbf{E}(x, y, z)$, $z \neq 0$. Suggestion: Apply Gauss' Law to the cylindrical surface S drawn in the picture below, and note that by symmetry, $\mathbf{E}$ must have the same magnitude and opposite direction at the top and bottom of the cylinder.

   (continued on other side)



2. Suppose that we have a spherically symmetric charge distribution; in particular, suppose that (in spherical coordinates)

$$\delta(\rho, \varphi, \theta) = \begin{cases} 3\rho & \text{if } \rho \leq 1, \\ 0 & \text{if } \rho > 1. \end{cases}$$

Use Gauss' Law and the fact about surface integrals to calculate $\mathbf{E}(x, y, z)$. Suggestion: Apply Gauss' Law to a sphere of radius R centered at the origin, and note that by symmetry, $\mathbf{E}$ must be pointed in the radially outward direction, with magnitude a function only of ρ (in spherical coordinates). Then there are two cases: $R \leq 1$ and $R > 1$.