Paragraph HW 05
Gradient search
Math 112, Spring 2006

Gradient search is a method for looking for a maximum value of a function $f(x, y)$ (or more generally, a function $f : \mathbb{R}^n \to \mathbb{R}$). In most cases, this method will produce at least a local maximum of $f(x, y)$, but it will not always produce a global maximum of $f(x, y)$.

In this homework, we will use the following version of gradient search.

The Gradient Search Algorithm. Suppose we want to find a local maximum of $f(x, y)$ that is near the point (a, b). We proceed as follows, starting with $(x, y) = (a, b)$.

1. Let $\mathbf{v} = \nabla f(x, y)$. Note that if $\nabla f(x, y) = \mathbf{0}$, we are at a critical point, which can be analyzed appropriately, and we are done. Otherwise, if we leave (x, y) in the direction of $\mathbf{v}$, f should increase, at least at first.

2. Leave (x, y) and go in the direction of $\mathbf{v}$ until f stops increasing. This will happen at a point where either $\nabla f(x, y) = \mathbf{0}$ or $\nabla f(x, y)$ is perpendicular to the path on which we are travelling (i.e., perpendicular to $\mathbf{v}$).

3. Let $\mathbf{v} = \nabla f(x, y)$ at our new point (x, y). Again, if $\mathbf{v} = \mathbf{0}$, then we are at a critical point (which we hope is a local maximum), and we are done. Otherwise, go back to step 2, to try to increase f again.

   (Exercise for CS majors: Rewrite this algorithm so that there are only two steps.)

So you can get a feel for what goes on with gradient search, in this homework, you will work through the algorithm by hand, but only approximately, using pictures. Specifically, the Maple worksheet for this assignment deals with two functions:

$$f(x, y) = 5e^{-x^2-y^2} + 3e^{-(x-5)^2-y^2},$$
$$g(x, y) = 5e^{-x^2-y^2} - 5e^{-(x-5)^2-y^2}.$$

The worksheet draws the graph of f, draws the contour diagram of f, and plots ∇f on the same axes as the contour diagram of f. (See below for how this is done.) It also does the same for g. Use these plots, especially the gradient/contour plots, to answer the following questions.

1. Suppose we apply gradient search to the function $f(x, y)$, starting at $(a, b) = (0.8, 0.8)$. Which critical point does the algorithm end up finding? Show your steps and briefly explain.

2. Find a different starting point (a, b) that leads gradient search to end up at a critical point that is not where the search in 1 ended up.

3. Using the example of $f(x, y)$, explain why gradient search does not always find the global maximum of a function, even when it finds a local maximum.

4. Now consider the function $g(x, y)$. If we start at $(a, b) = (0.8, 0.8)$, which critical point do we end up locating? What if we start at $(a, b) = (4, -0.2)$? Show your steps and briefly explain.

5. Find a starting point (a, b) such that if we apply gradient search to $g(x, y)$, starting at (a, b), the algorithm never finishes (never terminates). Briefly explain your answer.

Maple: Plotting vector fields

Code for this assignment. As usual, you can get a copy of the worksheet for this assignment either from the handouts folder, if you're working in the Math Lab, or from the course website:

`http://www.math.sjsu.edu/~hsu/courses/112/Math112-p05.mw`

Vector fields. In Maple, a vector field in $\mathbb{R}^2$ is represented by a pair of functions. For example, the gradient vector field $\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ is represented by:

```
> [diff(f(x,y),x),diff(f(x,y),y)];
```

In other words, $\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ is represented by the matrix $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$. Similarly, a vector field in $\mathbb{R}^3$ is represented by a matrix of three functions.

Plotting a vector field. To plot the gradient of $f(x, y)$ for $-3 \le x \le 7$ and $-2 \le y \le 2$, use the command:

```
> fieldplot([diff(f(x,y),x),diff(f(x,y),y)],x=-3..7,y=-2..2);
```