Notes on differentiability
Math 112

Given a function $f(x, y)$, how can we figure out if f is differentiable at (a, b)?

Sufficient conditions and their use. A *sufficient* condition for differentiability is one that allows you to conclude that f is differentiable. In other words, the logic goes, "If f satisfies the condition, then f is differentiable."

The most commonly used condition is:

Theorem. *If the partial derivatives of f exist and are continuous at every point in some neighborhood of (a, b), then f is differentiable at (a, b).*

Corollary. *If the partial derivatives of f exist and are continuous for all values of x and y, then f is differentiable for all values of x and y.*

In other words, "nice" formulas with "nice" partial derivatives are differentiable.

Necessary conditions and their use. A *necessary* condition for differentiability is something that you can conclude about f if you already know f is differentiable. In other words, the logic goes, "If f is differentiable, then f satisfies the condition." In the current context, such a condition is mostly useful for concluding that f is *not* differentiable. That is, we use the reasoning, "If f does *not* satisfy the condition, then f *cannot* be differentiable." (This alternate form of the same logic is called the *contrapositive* of the original statement.)

We list all of our necessary conditions in the following theorem.

Theorem. *If f is differentiable at (a, b), then:*

1. *f is continuous at (a, b);*
2. *The partial derivatives $f_x(a, b)$ and $f_y(a, b)$ exist;*
3. *For any unit vector $\mathbf{u}$, the directional derivative $\left.\dfrac{d}{dt}f((a, b) + t\mathbf{u})\right|_{t=0}$ exists and is given by the formula*

$$\left.\frac{d}{dt}f((a, b) + t\mathbf{u})\right|_{t=0} = \operatorname{grad} f(a, b) \cdot \mathbf{u}.$$

Taking the contrapositive, we get:

Corollary. *If **any** of the following is true, then f is not differentiable at (a, b).*

1. *f is not continuous at (a, b).*
2. *The partial derivative $f_x(a, b)$ does not exist.*
3. *The partial derivative $f_y(a, b)$ does not exist.*
4. *For some unit vector $\mathbf{u}$, the directional derivative $\left.\dfrac{d}{dt}f((a, b) + t\mathbf{u})\right|_{t=0}$ does not exist.*
5. *For some unit vector $\mathbf{u}$, the directional derivative $\left.\dfrac{d}{dt}f((a, b) + t\mathbf{u})\right|_{t=0}$ is not given by the formula*

$$\left.\frac{d}{dt}f((a, b) + t\mathbf{u})\right|_{t=0} = \operatorname{grad} f(a, b) \cdot \mathbf{u}.$$

How to use the full definition. If f does not satisfy some condition that is sufficient to conclude differentiability, but on the other hand satisfies all of the conditions necessary for differentiability, then we're stuck with trying to apply the full formal definition (p. 133 or p. 134). To apply this definition, we must essentially figure out a formal proof that the limit on p. 133 or p. 134 exists, and this is beyond the scope of our class.

Note, however, that we can use a computer to gather the following evidence for or against differentiability.

- If f is differentiable at (a, b), then its graph should look increasingly flat as you "zoom in" to (a, b). Otherwise, its graph should continue to look non-flat as you "zoom in."

- If f is differentiable at (a, b), then tables of its values as you "zoom in" to (a, b) should look increasingly linear. Otherwise, the tables should continue to look non-linear as you "zoom in."

While neither of these experiments gives a proof of differentiability or non-differentiability, they can give you a pretty good idea of the differentiability of f at (a, b), and they can serve as useful guides for what to try to prove.

An important example, revisited. Let

$$f(x, y) = \begin{cases} 1 & \text{if } 0 < y < x^2; \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the contour diagram of f is:
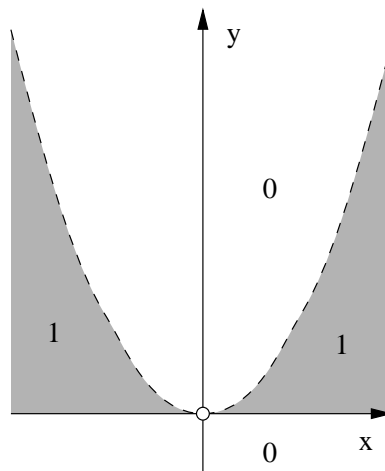
Here, the shaded region is where $f(x, y) = 1$, and the unshaded region (including the curve $y = x^2$ and the entire x-axis) is where $f(x, y) = 0$.

As before, note that if we approach the origin along *any* straight-line path, then the value of $f(x, y)$ will eventually become 0 at some distance away from the origin, and stay that way. Therefore, for any unit vector $\mathbf{u}$, $\left. \dfrac{d}{dt} f((0, 0) + t\mathbf{u}) \right|_{t=0} = 0$. However, as noted in the handout on continuity, f is not even continuous, so it is certainly not differentiable. Once again, the point is that our initial strategy of examining change one variable at a time, or even along arbitrary straight lines, fails in certain pathological situations; it is only the examination of change along arbitrary paths that gives us a complete understanding of continuity and differentiability.