Sample Exam 2
Math 108, Spring 2016

1. (12 points) Let $f : X \to Y$ be a function. State what the Inverse Theorem tells you about f.

2. (14 points) Define a relation $\sim$ on $\mathbf{R}^2$ by declaring that

$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

You are given that $\sim$ is an equivalence relation (i.e., do not spend time trying to prove that).

(a) Give a geometric description of $E_{(3,4)}$, the equivalence class of $(3, 4)$ under $\sim$.

(b) Give a geometric description/picture of a typical equivalence class under $\sim$, and describe any special cases (non-typical equivalence classes).

In questions 3–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (12 points) If S is a nonempty subset of $\mathbf{R}$, and u is an upper bound for S, then $u \in S$.

4. (12 points) The relation $f : \mathbf{Z} \to \mathbf{Z}$ given by:

$$f(x) = y \text{ if and only if } y^2 = x$$

is a well-defined function.

5. (16 points) **PROOF QUESTION.** Define a relation $\sim$ on $\mathbf{R}^2$ by declaring that

$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } x_2 - x_1 \in \mathbf{Z}.$$

Prove that $\sim$ is an equivalence relation.

6. (16 points) **PROOF QUESTION.** Let

$$X = \{x \in \mathbf{R} \mid x \geq 0\},$$
$$Y = \{y \in \mathbf{R} \mid y \geq 3\}.$$

Define a function $g : X \to Y$ by $g(x) = x^2 + 3$. (You may assume that g is well-defined.) Prove that g is a bijection.

7. (18 points) **PROOF QUESTION.** Let

$$S = \left\{ \frac{7n - 5}{2n} \,\middle|\, n \in \mathbf{Z},\ n > 0 \right\}.$$

Prove that $\sup S = \dfrac{7}{2}$.