Math 108, problem set 08
Outline due: Wed Apr 13
Completed version due: Mon Apr 18
Last revision due: Mon May 16

Exercises (to be done but not turned in): 18.3, 18.4, 18.5, 18.7, 19.2, 19.3, 19.4, 19.6.

Definitions:

union, intersection If X is a set, and A_i is a subset of X for all positive integers i, for $n \geq 1$, we define the union $\displaystyle\bigcup_{i=1}^{n} A_i$ recursively by

$$\bigcup_{i=1}^{1} A_i = A_1, \qquad \bigcup_{i=1}^{n+1} A_i = \left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}$$

The intersection $\displaystyle\bigcap_{i=1}^{n} A_i$ is defined analogously.

Problems to be turned in:

1. Prove, by induction on n, that for $n \in \mathbf{N}$ and $r \in \mathbf{R}$, we have $\displaystyle\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$.
2. Prove, by induction on n, that for $n \in \mathbf{N}$, $x \in \mathbf{R}$, $x > 0$, we have $(1 + x)^n \geq 1 + nx$.
3. Prove, by induction on n, that for $n \geq 1$, the set $\{1, \ldots, n\}$ has exactly 2^n subsets. (Suggestion: Every subset of $\{1, \ldots, n, n+1\}$ either contains the element $n + 1$ or it doesn't; how many subsets are there of each type?)
4. The goal of this problem is to prove that when we take a union of sets, we can modify those sets to be pairwise disjoint without changing their union.

   Specifically, let X be a set, and for each positive integer i, let A_i be a subset of X. For each positive integer i, define B_i by

   $$B_1 = A_1, \qquad B_i = A_i \Bigg\backslash \left(\bigcup_{k=1}^{i-1} A_k\right).$$

   (a) Prove that for $i, j \in \mathbf{Z}$, $1 \leq i < j$, $B_i \cap B_j = \emptyset$. (Suggestion: Proceed by contradiction, and note that $B_i \subseteq A_i$.)

   (b) Use induction on n and the inductive definition of the union of n sets (see above) to prove that $\displaystyle\bigcup_{k=1}^{n} A_k = \bigcup_{i=1}^{n} B_i$.

   (c) Prove that $\displaystyle\bigcup_{k=1}^{\infty} A_k = \bigcup_{i=1}^{\infty} B_i$. Suggestion: This does *not* follow directly from the previous part of the problem (why not?), though the previous part of the problem turns out to be useful.

(Continued on other side.)

5. Consider the function $g : \mathbf{R} \times \mathbf{N} \to \mathbf{R}$ defined (recursively) as follows.

$$g(b, n) = \begin{cases} 1 & \text{if } n = 0, \\ b \cdot g\left(b^2, \dfrac{n-1}{2}\right) & \text{if } n \text{ is odd}, \\ g\left(b^2, \dfrac{n}{2}\right) & \text{if } n \text{ is even}. \end{cases}$$

(a) Show the steps in the computation of $g(b, 13)$ for fixed $b \in \mathbf{R}$.

(b) Use (strong) induction on n to prove that if $b \in \mathbf{R}$ and $n \in \mathbf{N}$, then $g(b, n) = b^n$. You may freely use K-12 knowledge like the fact that every integer is either even or has the form $2k + 1$ for some integer k. (This is harder to prove than it looks — try it!)

6. 19.2.

7. 19.3.