Geometric Methods for Tracking Space Debris

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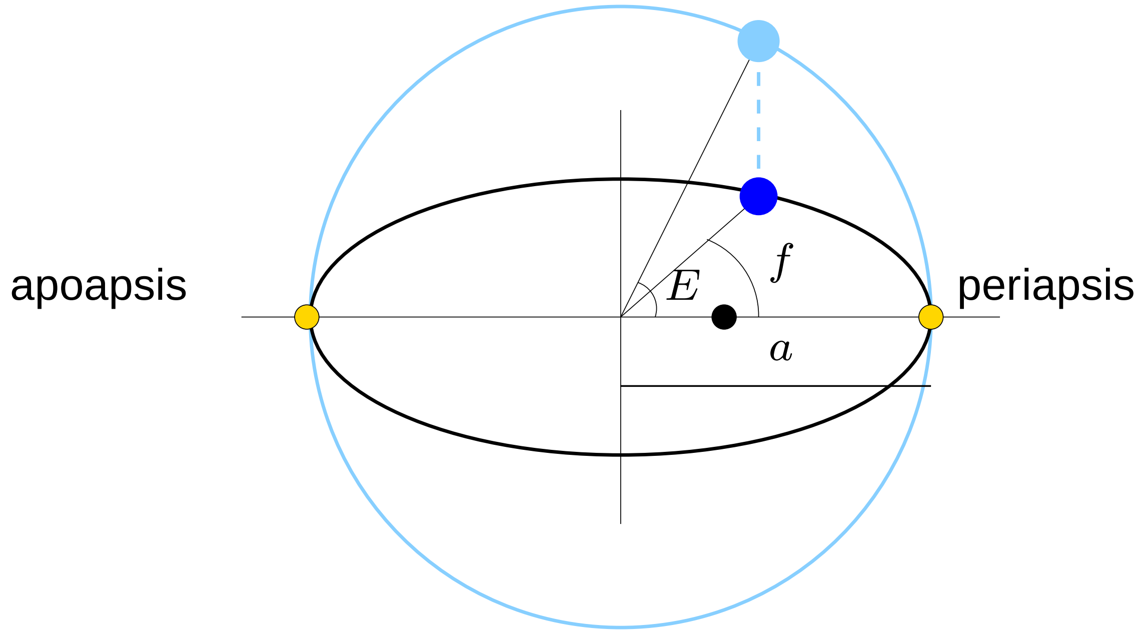
Problem Statement

- Given an observation of a previously uncorrelated space debris particle made by an optical telescope, what is the set of orbits that could have produced the given observation?
- Given a set of uncorrelated tracks, how can you identify the tracks corresponding to the same object?
- How can this process be automated to deal with correlation and orbit determination of many observations over a few week time period?

References

- G. Tommei, A. Milani, and A. Rossi, "Orbit Determination of Space Debris: Admissible Regions," *Celestial Mechanics and Dynamical Astronomy*, Vol. 97, pp. 289-304, 2007.
- J.M. Maruskin, D.J. Scheeres, K.T. Alfriend, "Correlation of Optical Observations of Objects in Earth Orbit," *Journal of Guidance, Control and Dynamics*, Vol. 32, pp. 57-85, 2009.
- J.M. Maruskin and D.J. Scheeres, "Metrics on the space of bounded Keplerian orbits and space situational awareness," *pre-print*, submitted to the *48th IEEE Conference on Decision and Control*, Shanghai, China, 2009.

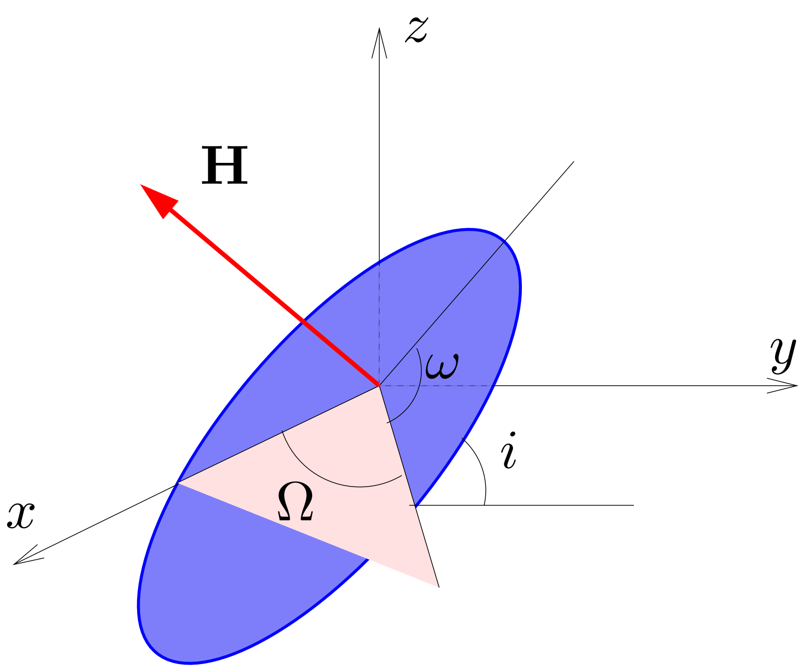
Anatomy of an Orbit



- E - eccentric anomaly.
- $\mu = GM$ - gravitational parameter.
- $n = \sqrt{\mu/a^3}$ - mean motion.
- $M = \begin{cases} E - e\sin E \\ n(t - t_p) \end{cases}$ - mean anomaly*.

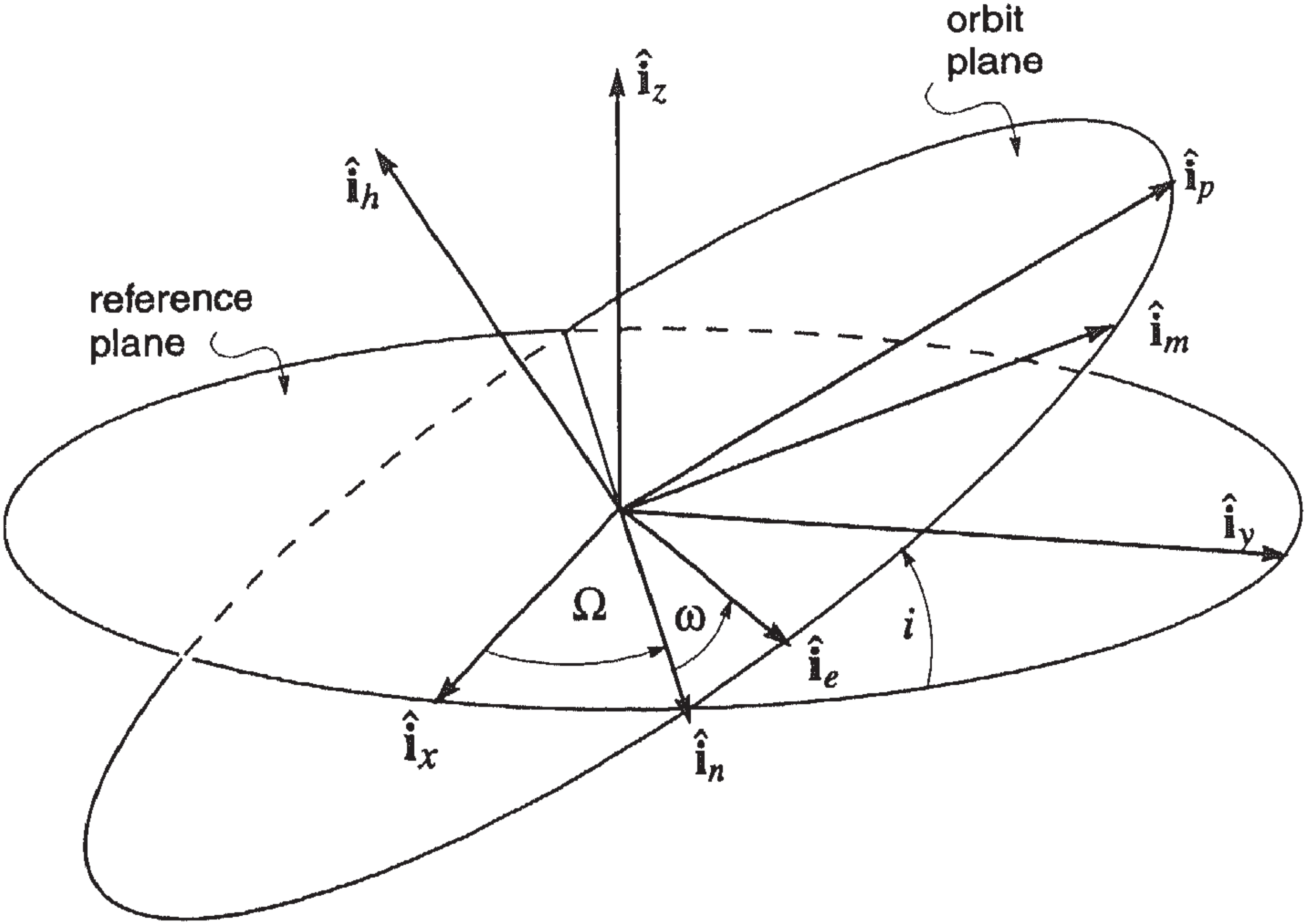
- e - eccentricity*.
- f - true anomaly.
- a - semi-major axis*.

The Anatomy of an Orbit



- Ω - Longitude of the Ascending Node*.
- ω - argument of periapsis*.
- i - inclination*.
- $\mathbf{H} = \mathbf{r} \times \mathbf{v}$ - angular momentum.

The Anatomy of an Orbit



The Attributable Vector

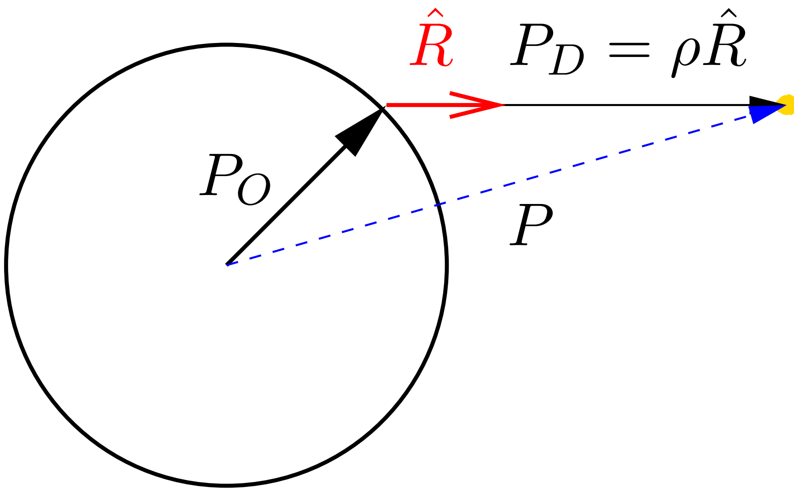
- An *optical attributable vector* is a vector

$$A = \langle \alpha, \delta, \dot{\alpha}, \dot{\delta} \rangle \in [-\pi, \pi) \times (-\pi/2, \pi/2) \times \mathbb{R}^2$$

observed at time t and observatory L (must be stored).

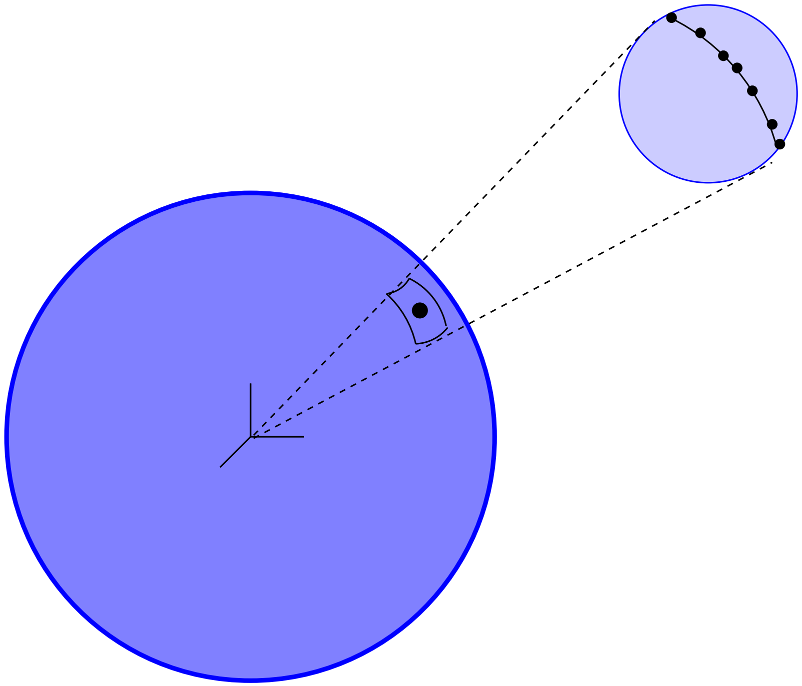
- Topocentric to geocentric coordinates:

$$P = P_O + P_D = P_O + \rho \hat{R}$$



Computation of the Attributable

- A *track* is an ordered set of topocentric angles $\{\alpha_i, \delta_i, t_i\}_{i=1}^{N}$



$$\alpha(t) = \alpha_0 + \dot{\alpha}_0(t - t_0) + \frac{1}{2}\ddot{\alpha}_0(t - t_0)^2 + \cdots$$

$$\delta(t) = \delta_0 + \dot{\delta}_0(t - t_0) + \frac{1}{2}\ddot{\delta}_0(t - t_0)^2 + \cdots$$

Computation of the Attributable

- We can concentrate the information spread out in the track to angle/angle-rates information at an epoch time t_0 within the track.
- We assume a precise knowledge of topocentric angles and angle-rates.
- We approximate the uncertainty distribution as a 2-D lamina in $(\rho, \dot{\rho})$ space, which gives rise to a 2-dimensional submanifold of phase space $\mathbb{R}^6$.
- The *extended attributable vector* $\mathfrak{X} = (\alpha, \delta, \dot{\alpha}, \dot{\delta}, \Phi, \Theta, h, t)$.

The Admissible Region

- Define the *specific geocentric energy* of the particle:

$$E = \frac{1}{2}||\dot{P}||^2 - \frac{\mu}{||P||}$$

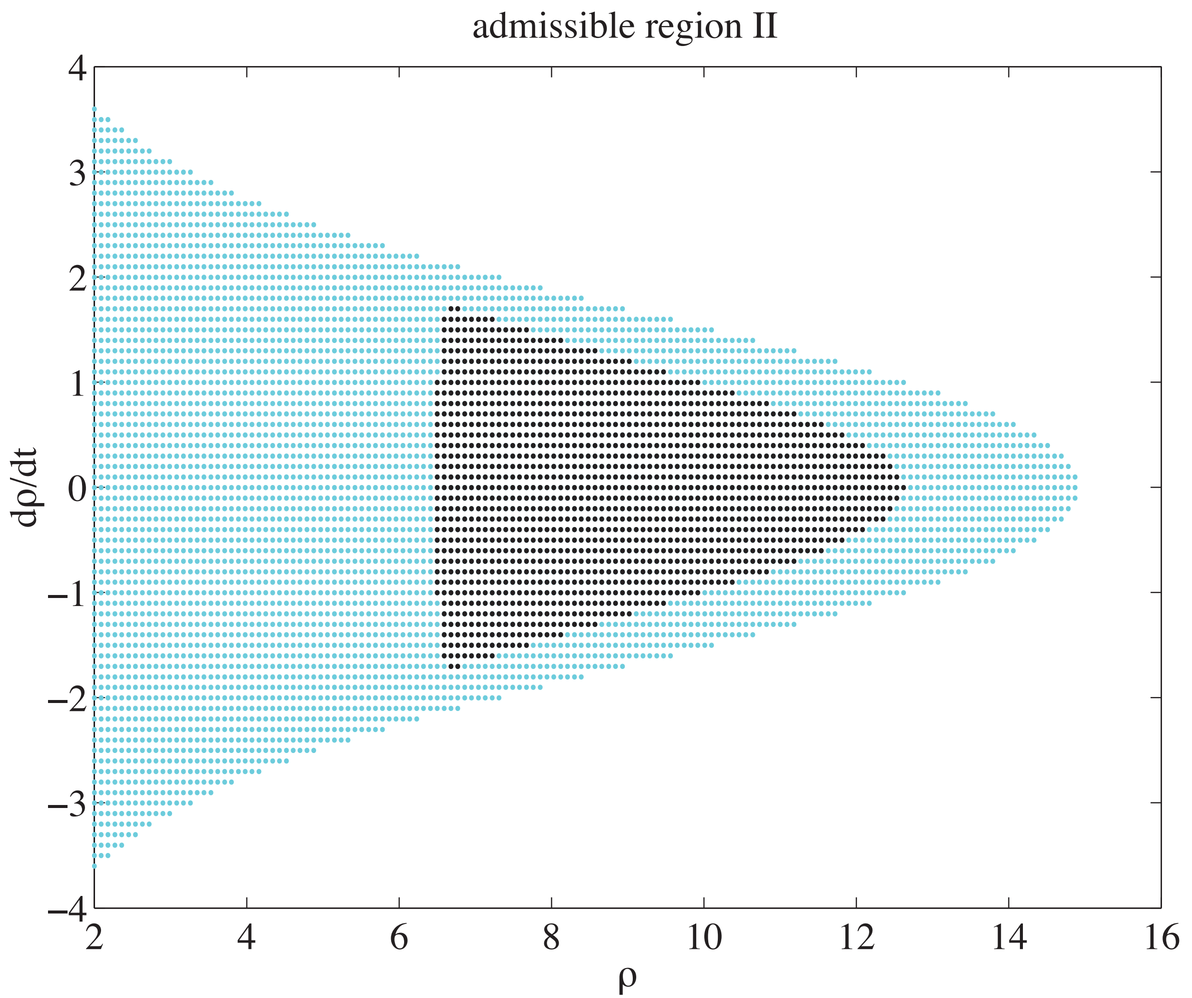
- Constraints on $(\rho, \dot{\rho})$ (Tommei, et al.):

$$\begin{aligned} C_1 &= \{(\rho, \dot{\rho}) : E < 0\} \\ C_2 &= \{(\rho, \dot{\rho}) : 2 < \rho < 20\} \\ C_3 &= \{(\rho, \dot{\rho}) : 1.03 < r_p\} \ \ (200 \text{ km}) \\ C_4 &= \{(\rho, \dot{\rho}) : r_a < 25\} \end{aligned}$$

- $\mathcal{E}_{\mathfrak{X}} = \bigcap C_i \subset \mathbb{R}^2$: *admissible region* of the $(\rho, \dot{\rho})$ plane.
- We populate the admissible region with *virtual debris* (VD) particles.

The Admissible Region

$$A = (0, \pi/6, 0.1, 0.03), \qquad \Phi = 0,\ \Theta = \pi/3$$



The Uncertainty Distribution

- The uncertainty distribution is a two-dimensional manifold, which can be easily parameterized by $\mathcal{C}$.
- In observation space, it is $\mathcal{C} \cup A$.
- In inertial geocentric cartesian coordinates it becomes more complicated, especially after time evolution.

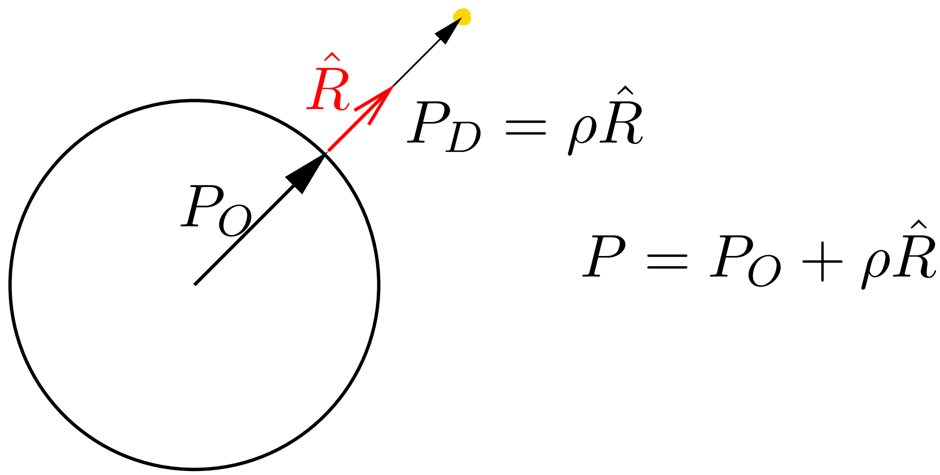
Delaunay Variables

- From the classical orbital elements $(a, e, i, \omega, \Omega, M)$ define:

$$l = M \qquad L = \sqrt{\mu a}$$
$$g = \omega \qquad G = L\sqrt{1 - e^2}$$
$$h = \Omega \qquad H = G \cos i$$

- These are the action/angle variables of the 2BP.
- These 6 Delaunay variables form symplectic pairs.
- The sum of the <u>signed</u> area projections is conserved.

Zenith Observations



- For a *zenith* observation, $P_0 || \hat{R}$. The *geocentric* angles are known and the geocentric radius r depends only upon ρ.
- Angular momentum is independent of $\dot{r}$, therefore also of $\dot{\rho}$.
- zenith obsservation $\Rightarrow G, h, H$ are independent of $\dot{\rho}$.

Equations of Motion

- Define Hamiltonian for the perturbed 2BP:

$$\mathcal{F} = -\frac{\mu^2}{2L^2} + \mathcal{R}(L, l, G, g, H, h)$$

- Delaunay variables evolve according to Hamilton's equations:

$$\frac{dl}{dt} = \frac{\partial \mathcal{F}}{\partial L} \qquad \frac{dL}{dt} = -\frac{\partial \mathcal{F}}{\partial l}$$

$$\frac{dg}{dt} = \frac{\partial \mathcal{F}}{\partial G} \qquad \frac{dG}{dt} = -\frac{\partial \mathcal{F}}{\partial g}$$

$$\frac{dh}{dt} = \frac{\partial \mathcal{F}}{\partial H} \qquad \frac{dH}{dt} = -\frac{\partial \mathcal{F}}{\partial h}$$

Kepler Problem

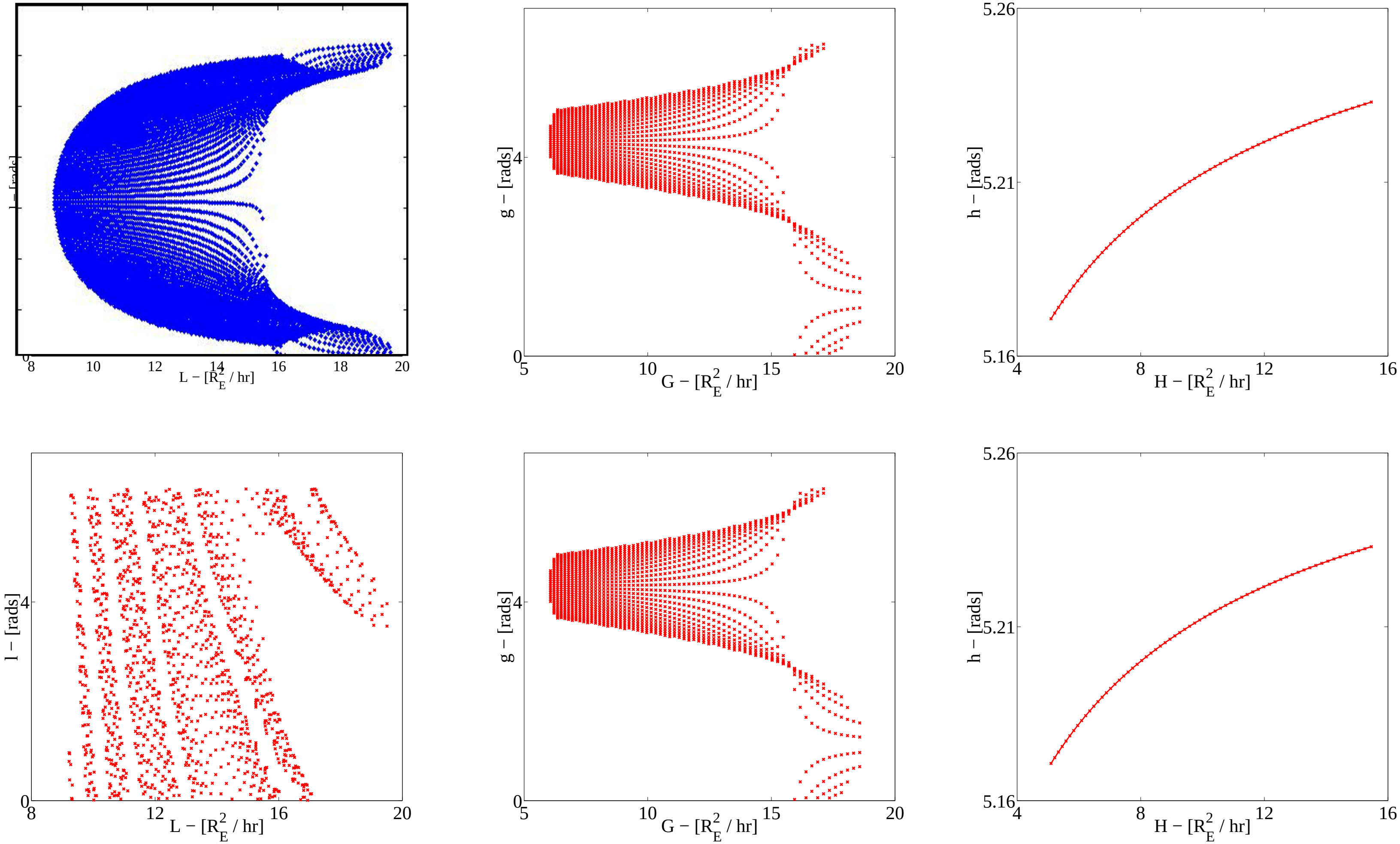
- Kepler Problem $\mathcal{F} = -\dfrac{\mu^2}{2L^2}$

$$\frac{dl}{dt} = \frac{\mu^2}{L^3}, \qquad \frac{dg}{dt} = \frac{dh}{dt} = \frac{dL}{dt} = \frac{dG}{dt} = \frac{dH}{dt} = 0.$$

- The (G, g) and (H, h) projections of the uncertainty distribution are constant.
- All of the VD dots “march” up the (L, l) plane at rates depending only upon L.
- Integral Invariants of Poincaré-Cartan $\Rightarrow$ projected area on (L, l) plane is conserved.
- $A = (0, \pi/6, 0.1, 0.03)$, $t = 0$, $\Theta = \pi/3$, $\Phi = 0$; zenith.

zenith observation/Kepler orbit

$$A = (0, \pi/6, 0.1, 0.03),\ \Theta = \pi/3,\ \Phi = 0, t = 0/70$$



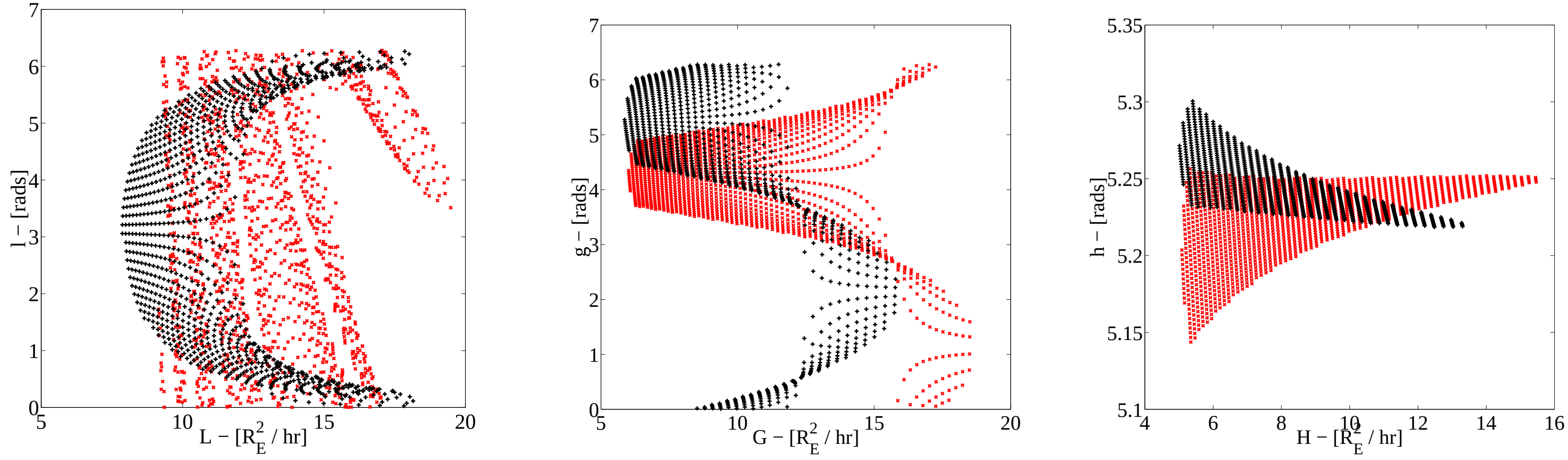
Intersection Theory Analysis (ITA)

- For pairwise UCTs, treat attributable vector A as determined (for now).
- Map Admissible Region to Delaunay variables.
- Project both surfaces onto the symplectic Delaunay planes.
- Choose one of the planes and cut off the nonoverlap portions of the surface.
- Do the same for a different plane.
- Repeat until a unique point or region emerges as victor.
- Only ever have to consider intersections of planar regions.

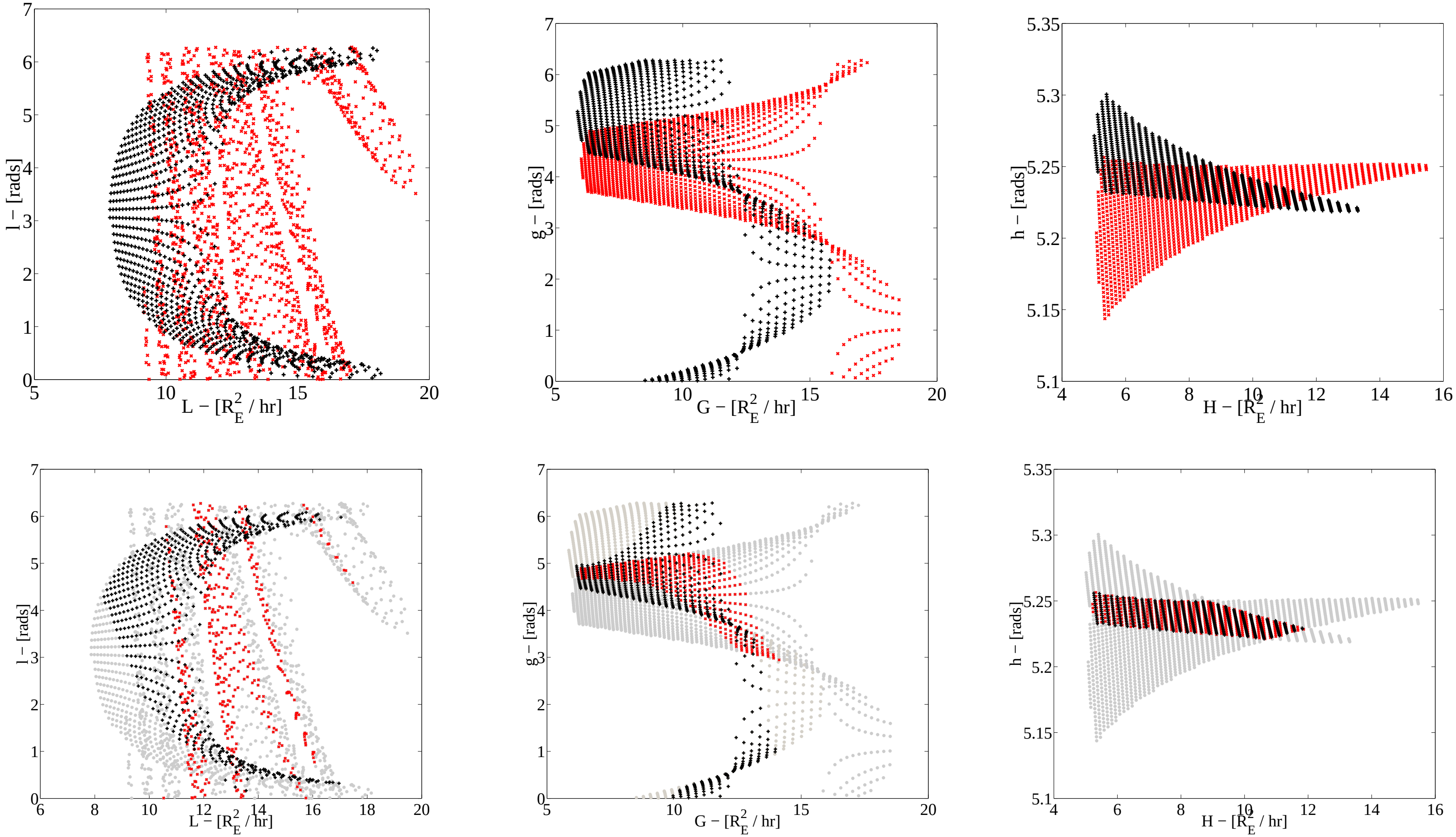
Non-zenith observations/Kepler orb

$A_1 = (0, \pi/6, 0.1, 0.03),\ \Theta_1 = \pi/3 + 0.1,\ \Phi_1 = 0.1,\ t = 0,$ red

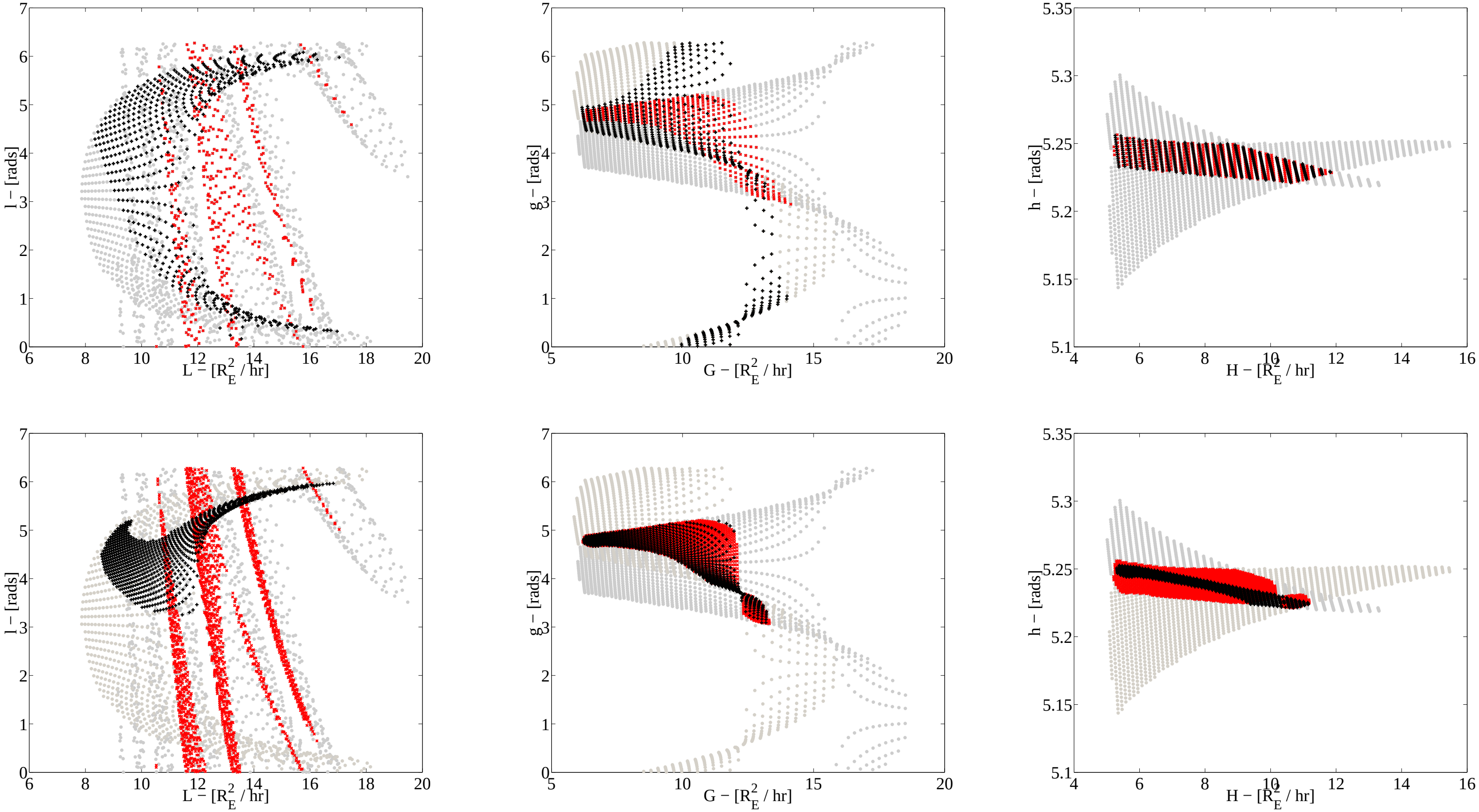
$A_2 = (1.152, 0.479, 0.226, -0.081),\ \Theta_2 = 1.252,\ \Phi_2 = 1.192, t = 70,$ blac



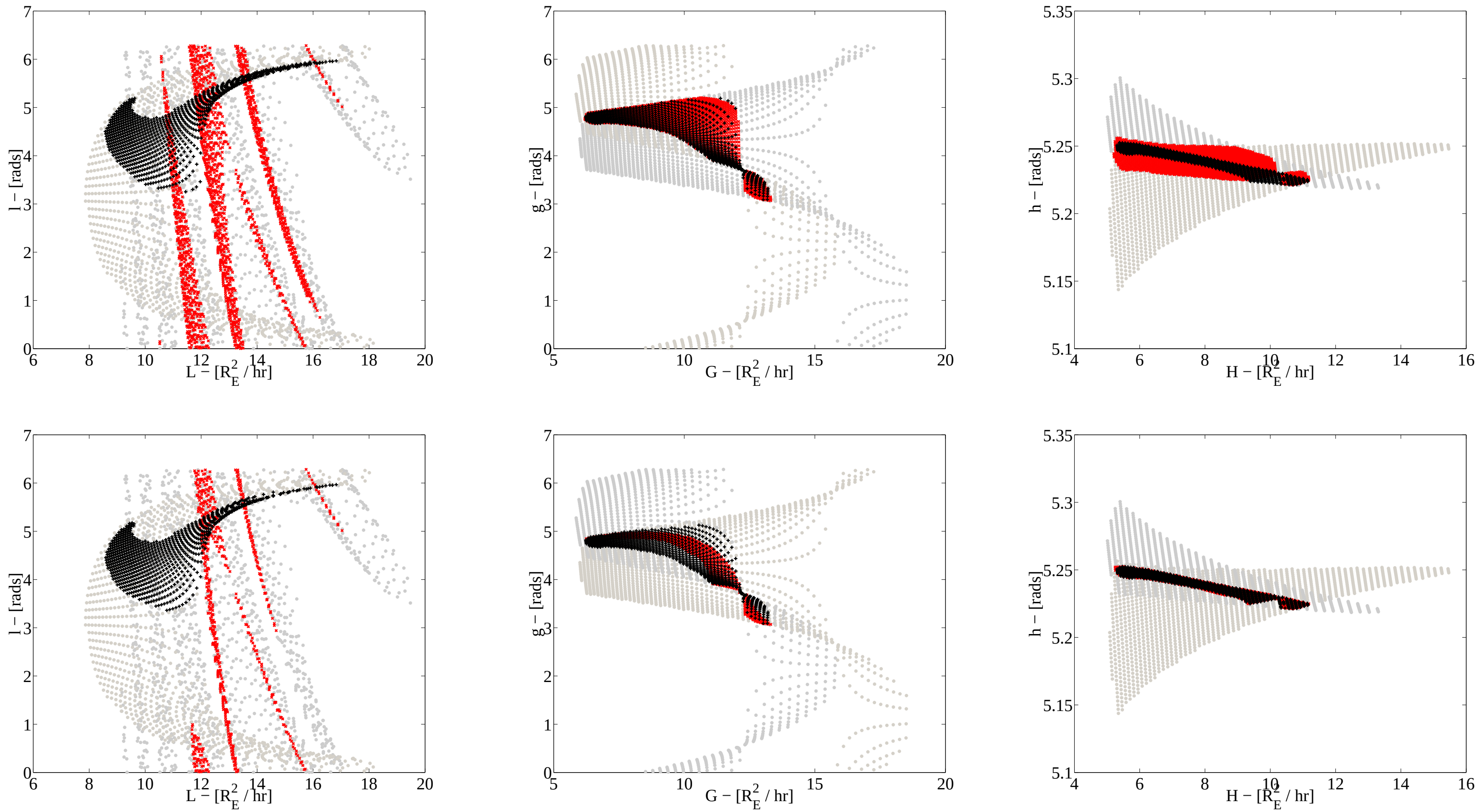
Non-zenith observations/Kepler orb



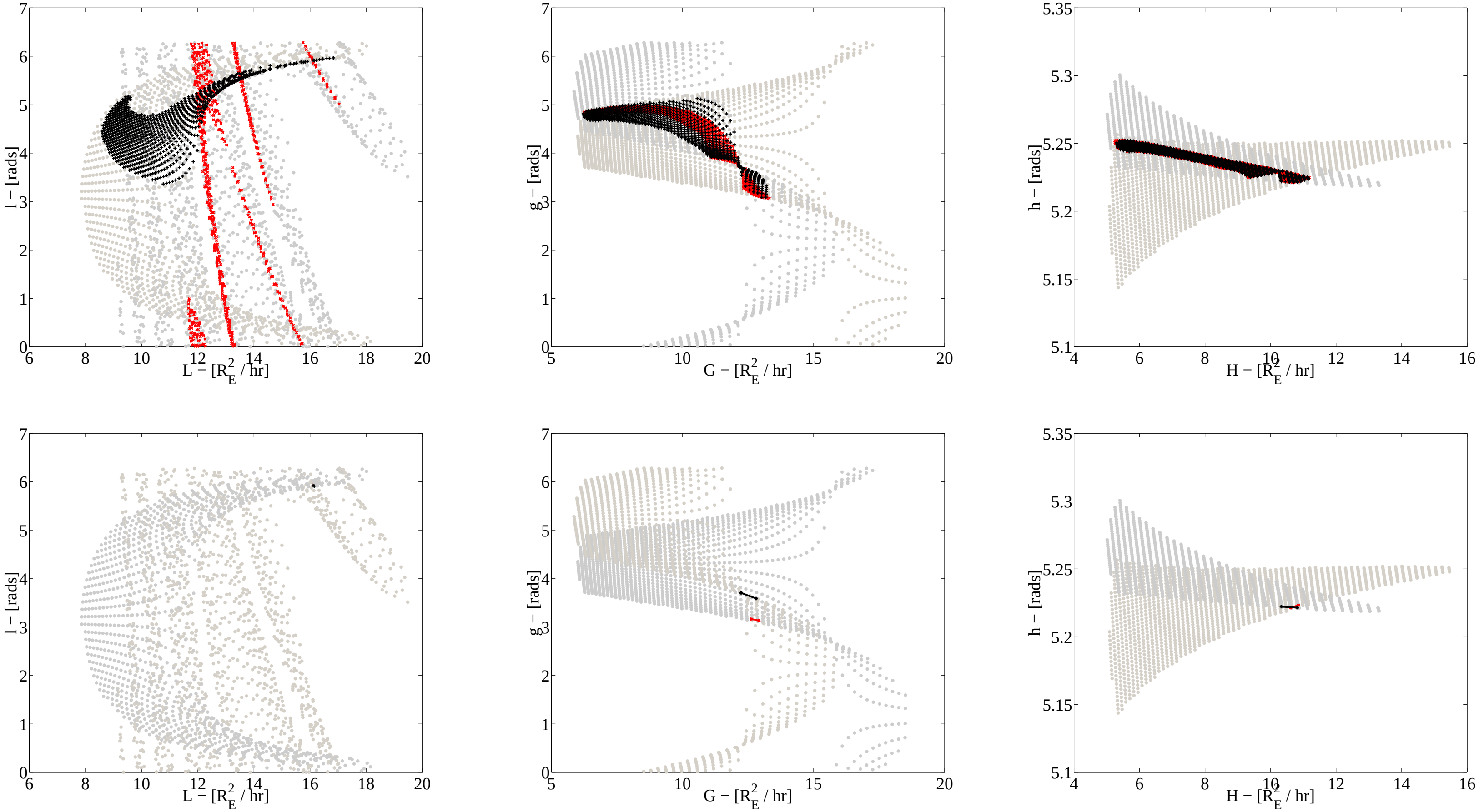
Non-zenith observations/Kepler orb



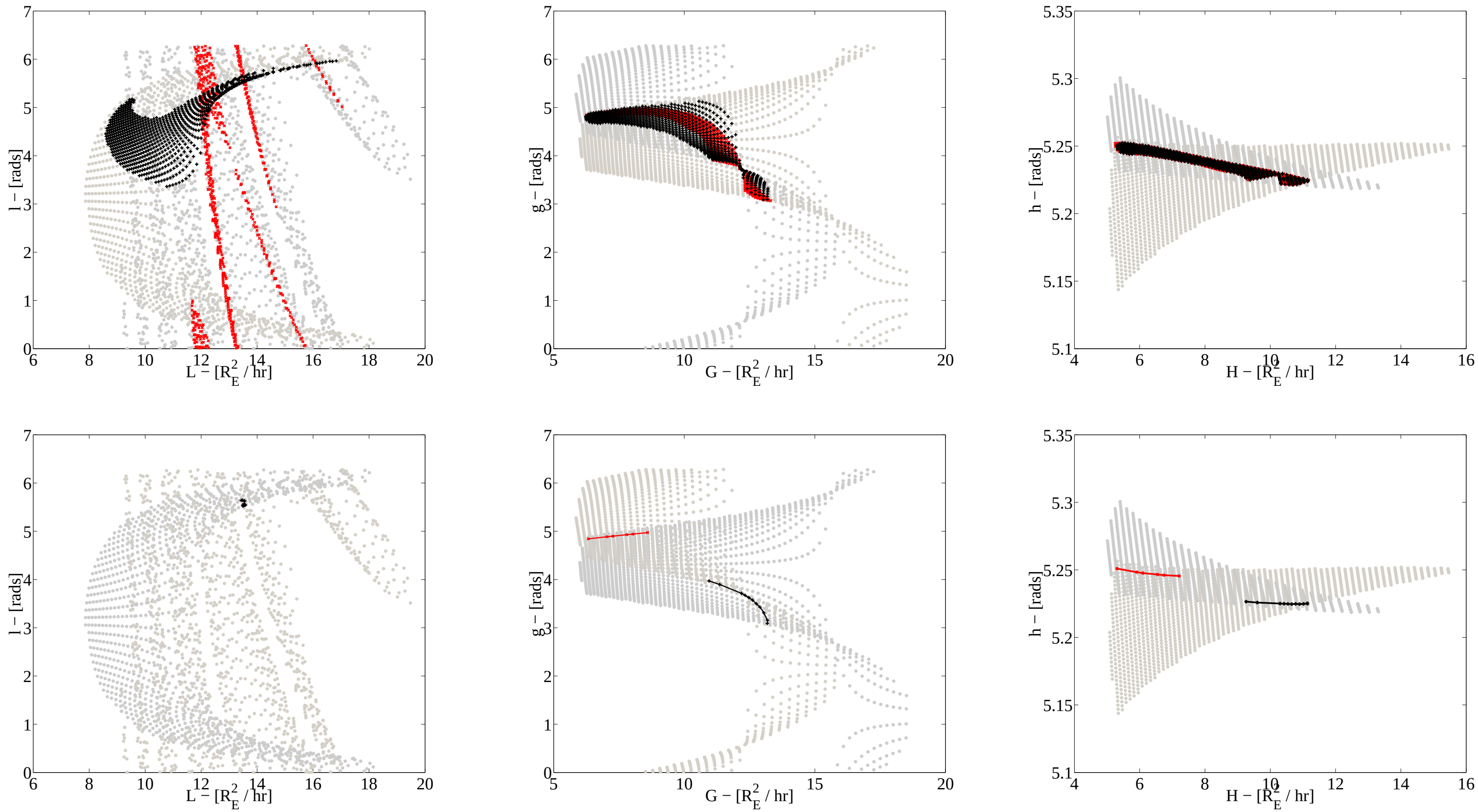
Non-zenith observations/Kepler orb



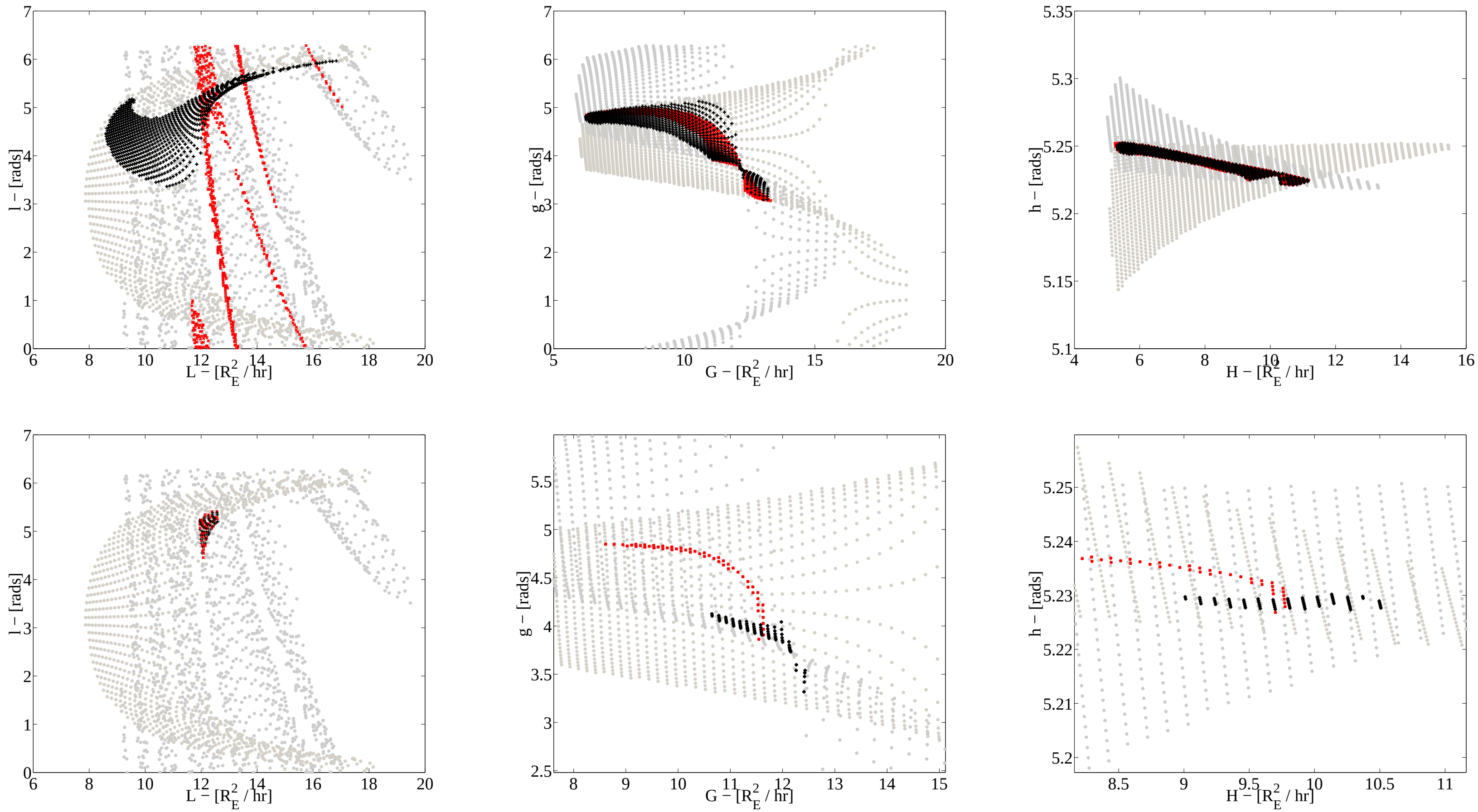
Non-zenith observations/Kepler orb



Non-zenith observations/Kepler orb



Non-zenith observations/Kepler orb



correlation Algorithm

- Record extended attributable vector for each uncorrelated track (UCT), i.e. $x = (A, t, L) \in \mathbb{R}^5 \times \mathbb{N}$, $x \to$ computer.
- Define a rolling observation window (eg. 1 week) and a central epoch time t_0 for that window.
- Compute uncertainty manifold for each UCT and map into Delaunay coordinates, then dynamically evolve or regress the distribution to the epoch time.
- Perform intersection theory analysis (ITA) for pairwise UCTs.
- Store hits in a temporary holding catalog for later confirmation.

The $\eta - \xi$ sphere

The $\mathbf{e}$ and $\mathbf{h}$ vectors.

- the eccentricity vector: $\mathbf{e} = \dfrac{\dot{\mathbf{r}} \times \mathbf{H}}{\mu} - \hat{\mathbf{r}} = e\hat{\mathbf{e}}$.
- angular momentum $\mathbf{H} = \mathbf{r} \times \dot{\mathbf{r}} = \sqrt{\mu a(1-e^2)}\hat{\mathbf{H}}$.
- normalized angular momentum: $\mathbf{h} = \dfrac{\mathbf{H}}{\sqrt{\mu a}} = \sqrt{1-e^2}\hat{\mathbf{H}}$.
- constraints: $\mathbf{e} \cdot \mathbf{e} + \mathbf{h} \cdot \mathbf{h} = 1$, i.e., $(\mathbf{e}, \mathbf{h}) \in S^5 \subset \mathbb{R}^6$.
- constraints: $\mathbf{e} \cdot \mathbf{h} = 0$.

The $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ vectors.

- $\boldsymbol{\eta} = \mathbf{e} + \mathbf{h}$.
- $\boldsymbol{\xi} = \mathbf{e} - \mathbf{h}$.
- constraints $\Longrightarrow \boldsymbol{\eta} \cdot \boldsymbol{\eta} = 1$ and $\boldsymbol{\xi} \cdot \boldsymbol{\xi} = 1$.
- $(\boldsymbol{\eta}, \boldsymbol{\xi}) \in S^2 \times S^2$, the space of isoergic, bound Keplerian orbits.

Mappings of AR to S^2

Recall

$$\mathbf{r} = \mathbf{r}_0 + \rho\hat{\mathbf{R}} \tag{1}$$

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_0 + \dot{\rho}\hat{\mathbf{R}} + \rho\dot{\hat{\mathbf{R}}}, \tag{2}$$

where

$$\dot{\hat{\mathbf{R}}} = \hat{\mathbf{R}}_\alpha\dot{\alpha} + \hat{\mathbf{R}}_\delta\dot{\delta}.$$

Mappings from the admissible region to S^2:

$$\mathrm{H}_{\mathfrak{x}} : \mathcal{E}_{\mathfrak{x}} \to S^2 \qquad \text{and} \qquad \Xi_{\mathfrak{x}} : \mathcal{E}_{\mathfrak{x}} \to S^2.$$

Zenith Observation

$\hat{\mathbf{R}} = \mathbf{r}$. Moreover, $\dot{\hat{\mathbf{R}}} \cdot \hat{\mathbf{R}} = 0$, $\dot{\mathbf{r}}_0 \cdot \mathbf{r}_0 = 0$, $\mathbf{r}_0 \cdot \dot{\hat{\mathbf{R}}} = 0$, and $\hat{\mathbf{R}} \cdot \dot{\mathbf{r}}_0 = 0$.

$$a(\rho, \dot{\rho}) = \frac{-\mu}{\frac{\pi^2}{144} + 2\rho\dot{\hat{\mathbf{R}}} \cdot \dot{\mathbf{r}}_0 + \rho^2 \dot{\hat{\mathbf{R}}} \cdot \dot{\hat{\mathbf{R}}} - \frac{2\mu}{1+\rho} + \dot{\rho}^2}.$$

Define:

$$\chi(\rho) = \left\| \dot{\mathbf{r}}_0 + \rho\dot{\hat{\mathbf{R}}} \right\| \quad \text{and} \quad \lambda(\rho) = \frac{(1+\rho)}{\mu}\chi^2(\rho) - 1$$

$$\sigma(\rho, \dot{\rho}) = \frac{-(1+\rho)\dot{\rho}}{\mu} \quad \text{and} \quad \tau(\rho, \dot{\rho}) = \frac{1+\rho}{\sqrt{\mu a}}$$

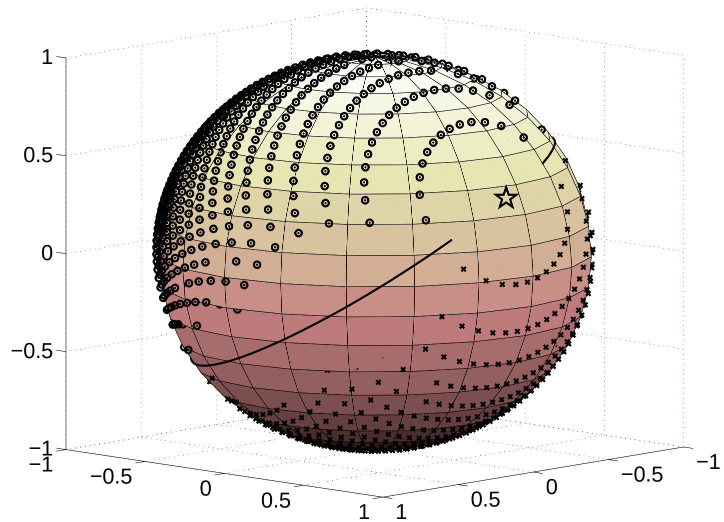
and the vectors

$$\mathbf{u} = \hat{\mathbf{R}} \quad \mathbf{v}(\rho) = \dot{\mathbf{r}}_0 + \rho\dot{\hat{\mathbf{R}}} \quad \mathbf{w}(\rho) = \hat{\mathbf{R}} \times \left(\dot{\mathbf{r}}_0 + \rho\dot{\hat{\mathbf{R}}} \right).$$

Zenith Observation

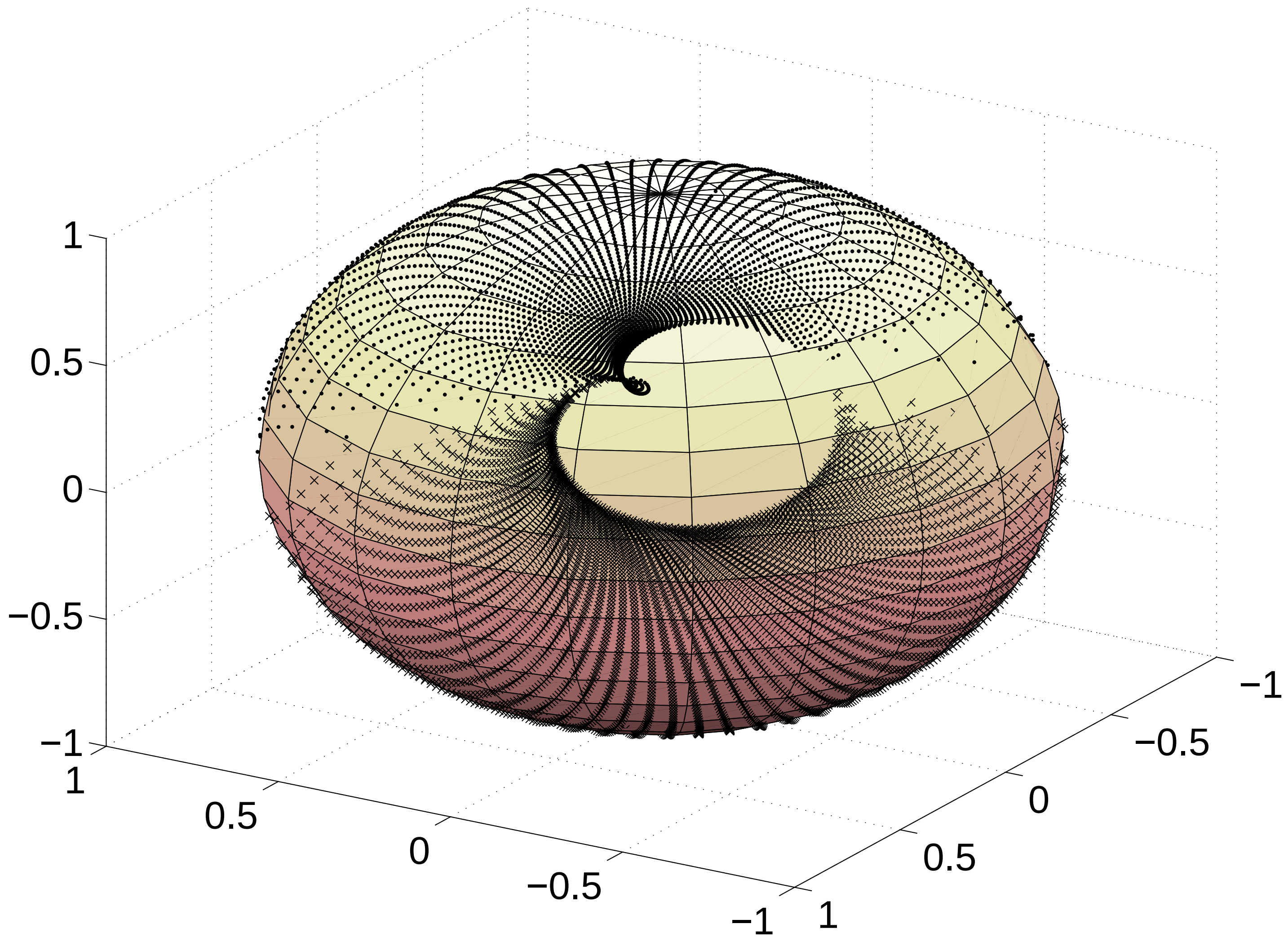
The vectors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ may be expressed as

$$\boldsymbol{\eta} = \lambda(\rho)\mathbf{u} + \sigma(\rho,\dot\rho)\mathbf{v}(\rho) + \tau(\rho,\dot\rho)\mathbf{w}(\rho)$$
$$\boldsymbol{\xi} = \lambda(\rho)\mathbf{u} + \sigma(\rho,\dot\rho)\mathbf{v}(\rho) - \tau(\rho,\dot\rho)\mathbf{w}(\rho).$$

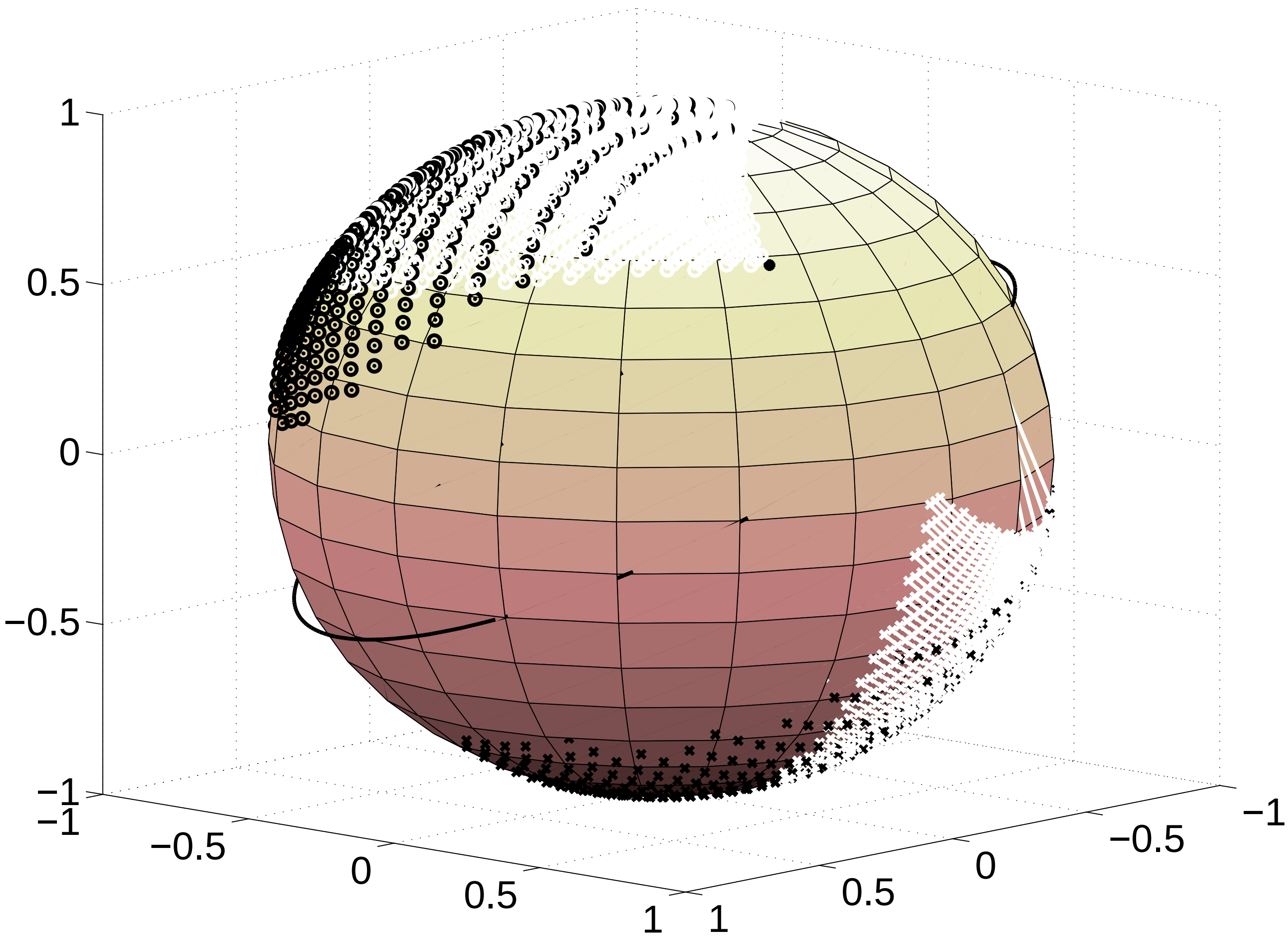


$\mathrm{H}_{\mathfrak{X}} : \mathcal{E}_{\mathfrak{X}} \to S^2$ and $\Xi_{\mathfrak{X}} : \mathcal{E}_{\mathfrak{X}} \to S^2$ are diffeomorphisms.

Regular Observations



Regular Observations



Metrics

- $\mathbb{V}(E)$- 4-d space of bounded, Keplerian orbits, $\mathbb{V}(E) \cong S^2 \times S^2$.
- For $\mathcal{O}_1, \mathcal{O}_2 \in \mathbb{V}(E)$, define a metric:

$$d^2(\mathcal{O}_1, \mathcal{O}_2) = ||\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2||^2 + ||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2||^2$$

- For two attributable vectors $\mathfrak{X}_1, \mathfrak{X}_2$, one can define a norm of the product space $\mathcal{E}_{\mathfrak{X}_1} \times \mathcal{E}_{\mathfrak{X}_2}$. Let $z \in \mathcal{E}_{\mathfrak{X}_1} \times \mathcal{E}_{\mathfrak{X}_2}$, then

$$D(z) = ||z||^2 = d^2(\rho_1, \dot{\rho}_1, \rho_2, \dot{\rho}_2).$$

- By following the *path of steepest descent*, one can find the closest point between the two manifolds with respect to this metric by solving

$$\frac{dz}{ds} = -\nabla D \qquad \text{with} \qquad z(0) = z_0,$$

and then taking

$$z^* = \lim_{s \to \infty} z(s).$$

Metric Gradient

The metric can be written as

$$D = 4 - 2\boldsymbol{\eta}_1 \cdot \boldsymbol{\eta}_2 - 2\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2.$$

Therefore the gradient is given by

$$\frac{\partial D}{\partial \rho_1} = -2\boldsymbol{\eta}_2 \cdot \frac{\partial \boldsymbol{\eta}_1}{\partial \rho_1} - 2\boldsymbol{\xi}_2 \cdot \frac{\partial \boldsymbol{\xi}_1}{\partial \rho_1} \tag{3}$$

$$\frac{\partial D}{\partial \dot{\rho}_1} = -2\boldsymbol{\eta}_2 \cdot \frac{\partial \boldsymbol{\eta}_1}{\partial \dot{\rho}_1} - 2\boldsymbol{\xi}_2 \cdot \frac{\partial \boldsymbol{\xi}_1}{\partial \dot{\rho}_1} \tag{4}$$

$$\frac{\partial D}{\partial \rho_2} = -2\boldsymbol{\eta}_1 \cdot \frac{\partial \boldsymbol{\eta}_2}{\partial \rho_2} - 2\boldsymbol{\xi}_1 \cdot \frac{\partial \boldsymbol{\xi}_2}{\partial \rho_2} \tag{5}$$

$$\frac{\partial D}{\partial \dot{\rho}_2} = -2\boldsymbol{\eta}_1 \cdot \frac{\partial \boldsymbol{\eta}_2}{\partial \dot{\rho}_2} - 2\boldsymbol{\xi}_1 \cdot \frac{\partial \boldsymbol{\xi}_2}{\partial \dot{\rho}_2}. \tag{6}$$

Metric Gradient

The partials are...

$$\frac{\partial \boldsymbol{\eta}}{\partial \rho} = \frac{1}{\mu}\left[v^2\hat{\mathbf{R}} - (\mathbf{v}\cdot\hat{\mathbf{R}})\mathbf{v}\right] - \frac{1}{r}\left[\hat{\mathbf{R}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{R}})\hat{\mathbf{r}}\right] + \frac{1}{\mu}\left[2(\mathbf{v}\cdot\dot{\hat{\mathbf{R}}})\mathbf{r} - (\mathbf{r}\cdot\dot{\hat{\mathbf{R}}})\mathbf{v} - (\mathbf{r}\cdot\mathbf{v})\dot{\hat{\mathbf{R}}}\right]$$
$$- \frac{1}{\sqrt{\mu a}}\mathbf{v}\times\hat{\mathbf{R}} - \frac{a}{r^3}(\mathbf{r}\cdot\hat{\mathbf{R}})\mathbf{h} + \frac{1}{\sqrt{\mu a}}\mathbf{r}\times\dot{\hat{\mathbf{R}}} - \frac{a}{\mu}(\mathbf{v}\cdot\dot{\hat{\mathbf{R}}})\mathbf{h} \quad (7)$$

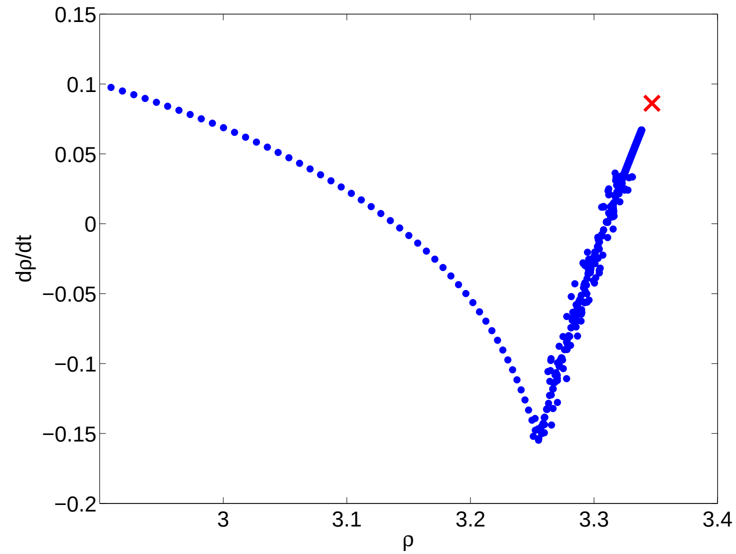
$$\frac{\partial \boldsymbol{\eta}}{\partial \dot{\rho}} = \frac{1}{\mu}\left[2(\mathbf{v}\cdot\hat{\mathbf{R}})\mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{R}})\mathbf{v} - (\mathbf{r}\cdot\mathbf{v})\hat{\mathbf{R}}\right] + \frac{1}{\sqrt{\mu a}}\mathbf{r}\times\hat{\mathbf{R}} - \frac{a}{\mu}(\mathbf{v}\cdot\hat{\mathbf{R}})\mathbf{h} \quad (8)$$

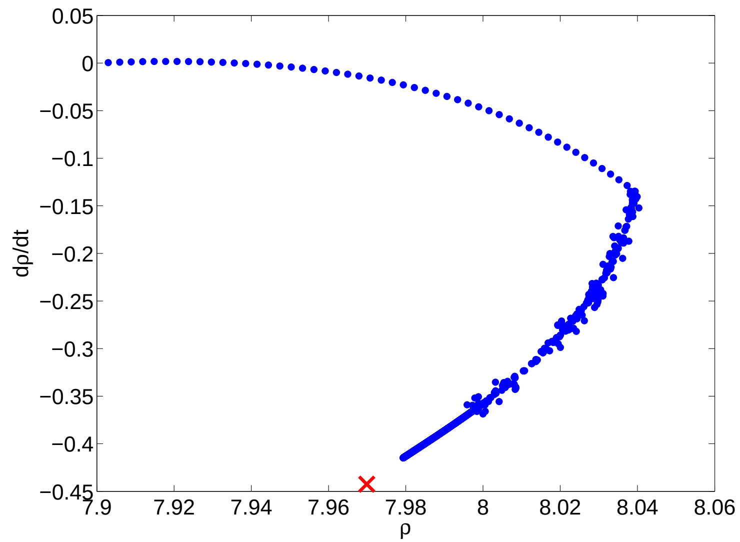
$$\frac{\partial \boldsymbol{\xi}}{\partial \rho} = \frac{1}{\mu}\left[v^2\hat{\mathbf{R}} - (\mathbf{v}\cdot\hat{\mathbf{R}})\mathbf{v}\right] - \frac{1}{r}\left[\hat{\mathbf{R}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{R}})\hat{\mathbf{r}}\right] + \frac{1}{\mu}\left[2(\mathbf{v}\cdot\dot{\hat{\mathbf{R}}})\mathbf{r} - (\mathbf{r}\cdot\dot{\hat{\mathbf{R}}})\mathbf{v} - (\mathbf{r}\cdot\mathbf{v})\dot{\hat{\mathbf{R}}}\right]$$
$$+ \frac{1}{\sqrt{\mu a}}\mathbf{v}\times\hat{\mathbf{R}} + \frac{a}{r^3}(\mathbf{r}\cdot\hat{\mathbf{R}})\mathbf{h} - \frac{1}{\sqrt{\mu a}}\mathbf{r}\times\dot{\hat{\mathbf{R}}} + \frac{a}{\mu}(\mathbf{v}\cdot\dot{\hat{\mathbf{R}}})\mathbf{h} \quad (9)$$

$$\frac{\partial \boldsymbol{\xi}}{\partial \dot{\rho}} = \frac{1}{\mu}\left[2(\mathbf{v}\cdot\hat{\mathbf{R}})\mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{R}})\mathbf{v} - (\mathbf{r}\cdot\mathbf{v})\hat{\mathbf{R}}\right] - \frac{1}{\sqrt{\mu a}}\mathbf{r}\times\hat{\mathbf{R}} + \frac{a}{\mu}(\mathbf{v}\cdot\hat{\mathbf{R}})\mathbf{h}, \quad (10)$$

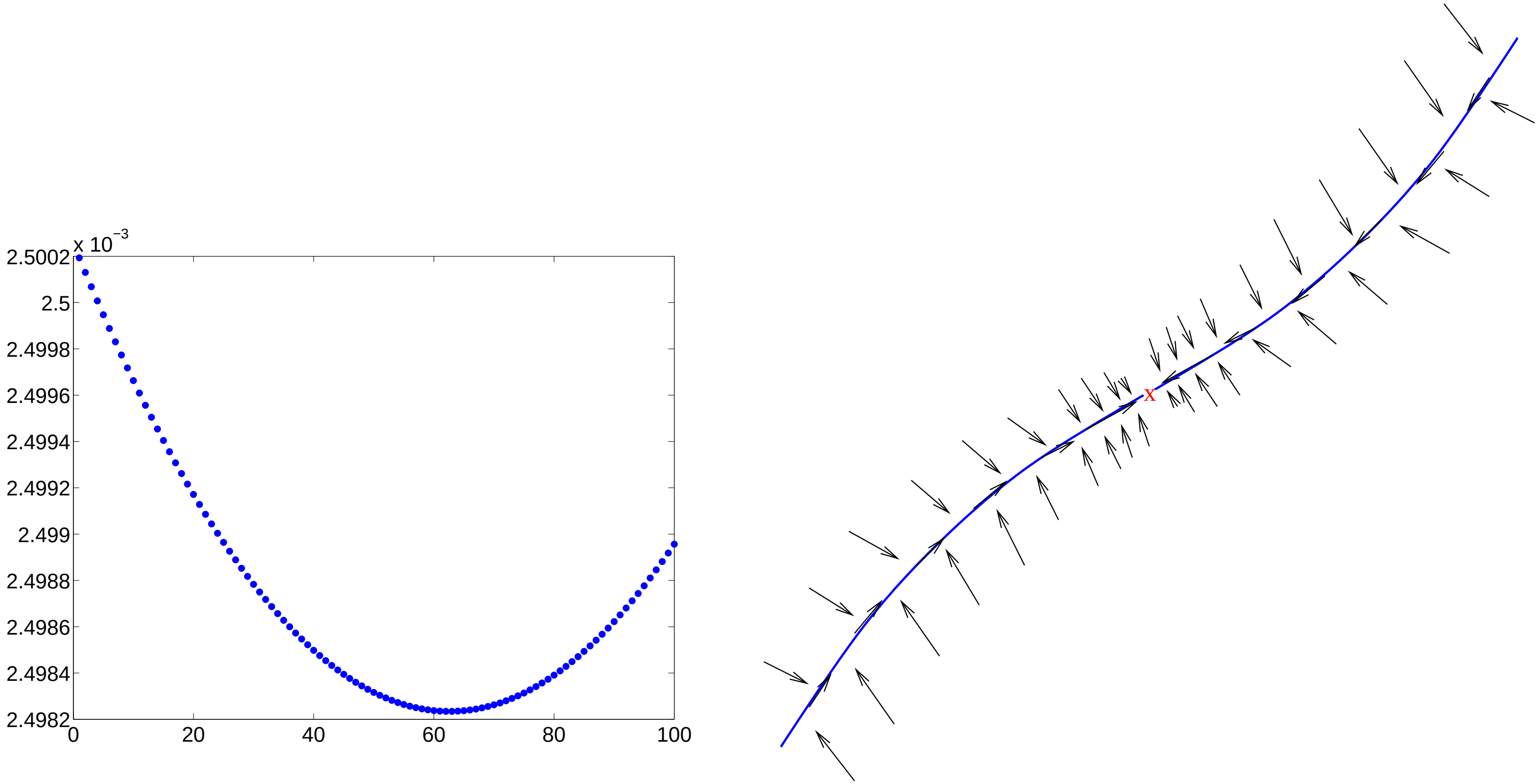
where a, $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{h}$ are functions of $(\rho, \dot{\rho})$ and everything is a function of the parameters $\alpha, \delta, \dot{\alpha}, \dot{\delta}$, Φ, Θ.

Numerical Implementation





Numerical Investigation



A “side view” of the trough (left).

An “overhead” view of the trough and gradient field (right).

Future Work

- Develop efficient computer algorithms to carry out intersection theory analysis on the $\eta - \xi$ spheres.
- Method for efficiently recognizing and “scaling” troughs in double-admissible regions.
- Treatment of “thick manifolds” using local linearization (STM, area expansion factors, etc.) about discretized points, for the purpose of handling uncertainties in the attributable vector $(\alpha, \delta, \dot{\alpha}, \dot{\delta})$.
- Develop technology to handle initial orbit determinations and correlations in the face of a large number of observations per night.