Geometry, Topology, Group Theory, & Killer Robots

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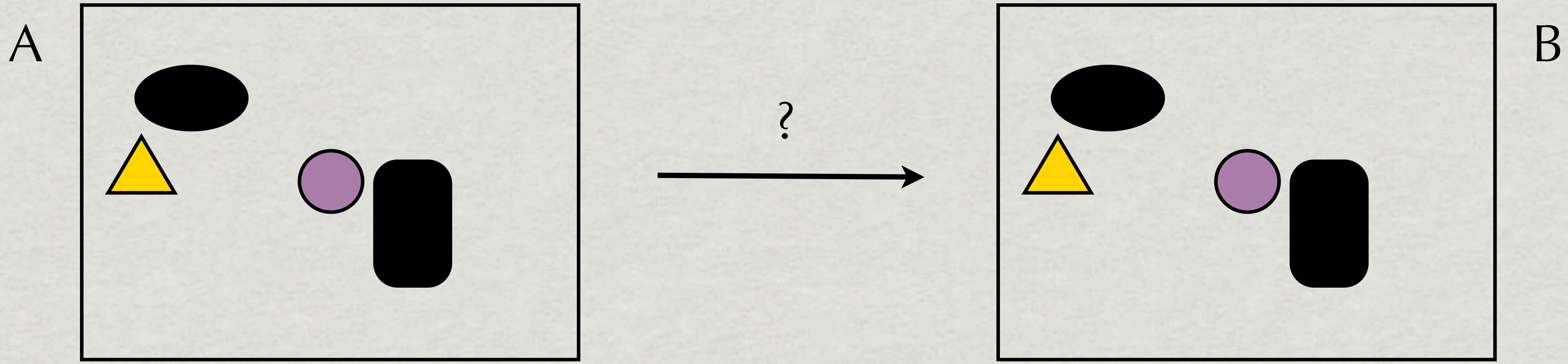
Chapter 1: Motivation

...from the world of manufacturing

Suppose you need to coordinate robotic agents moving on your factory floor.

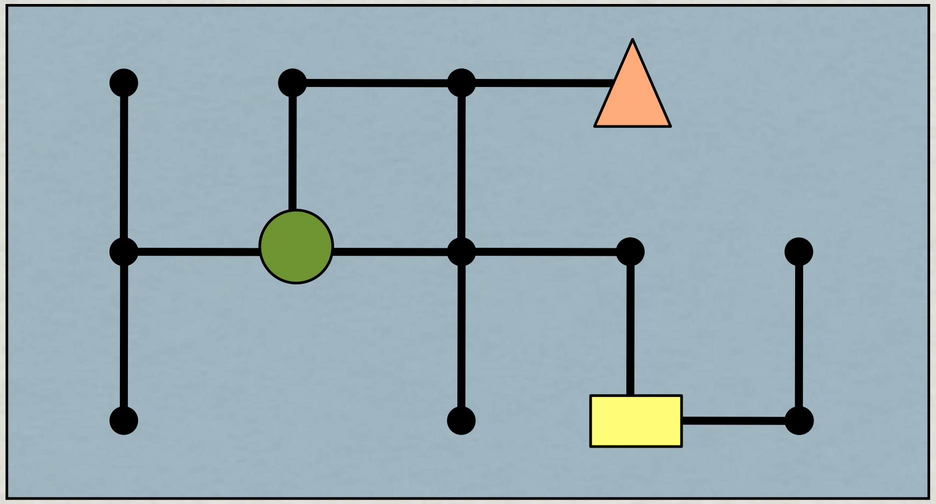


- Can I get from configuration A to configuration B *without collisions*?
- If so, how can I get from A to B *optimally*?

The Big Idea: We'll build a space that records the allowable positions of our system and then study that space.

Re:configuration

Often, constraints on our robots imply that the movements we wish to consider are *discrete*.

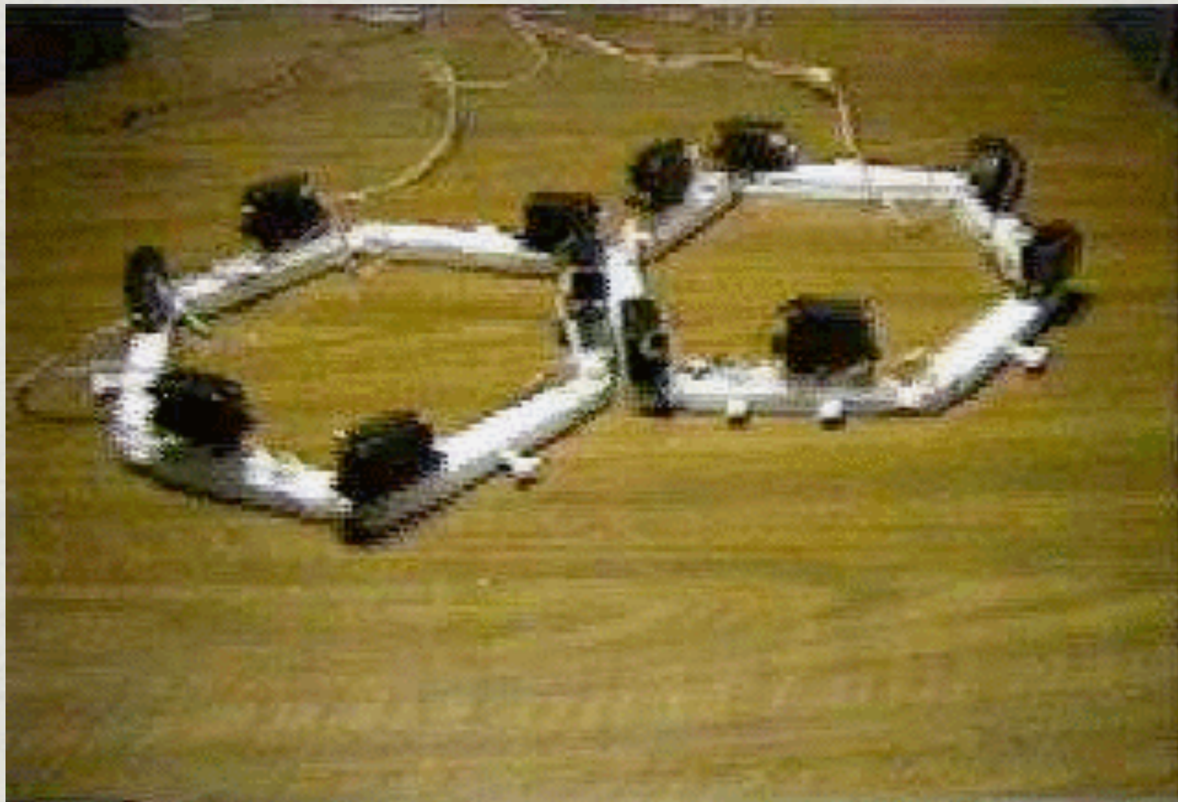


- tracks in the floor
- electrified guidewires
- optical paths

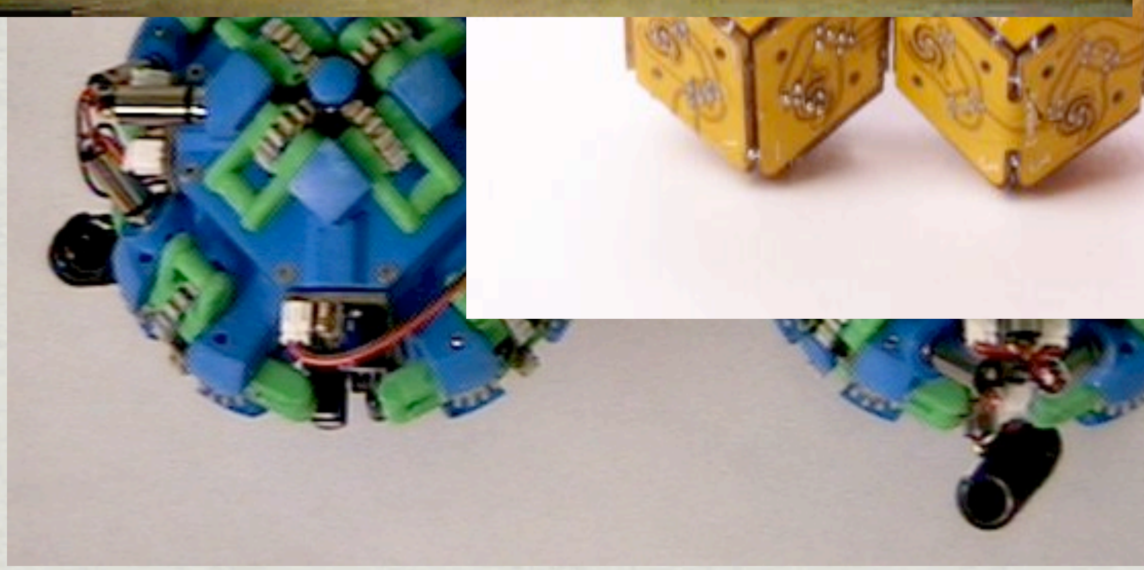
This discrete movement is what we refer to when using the term **reconfiguration**.

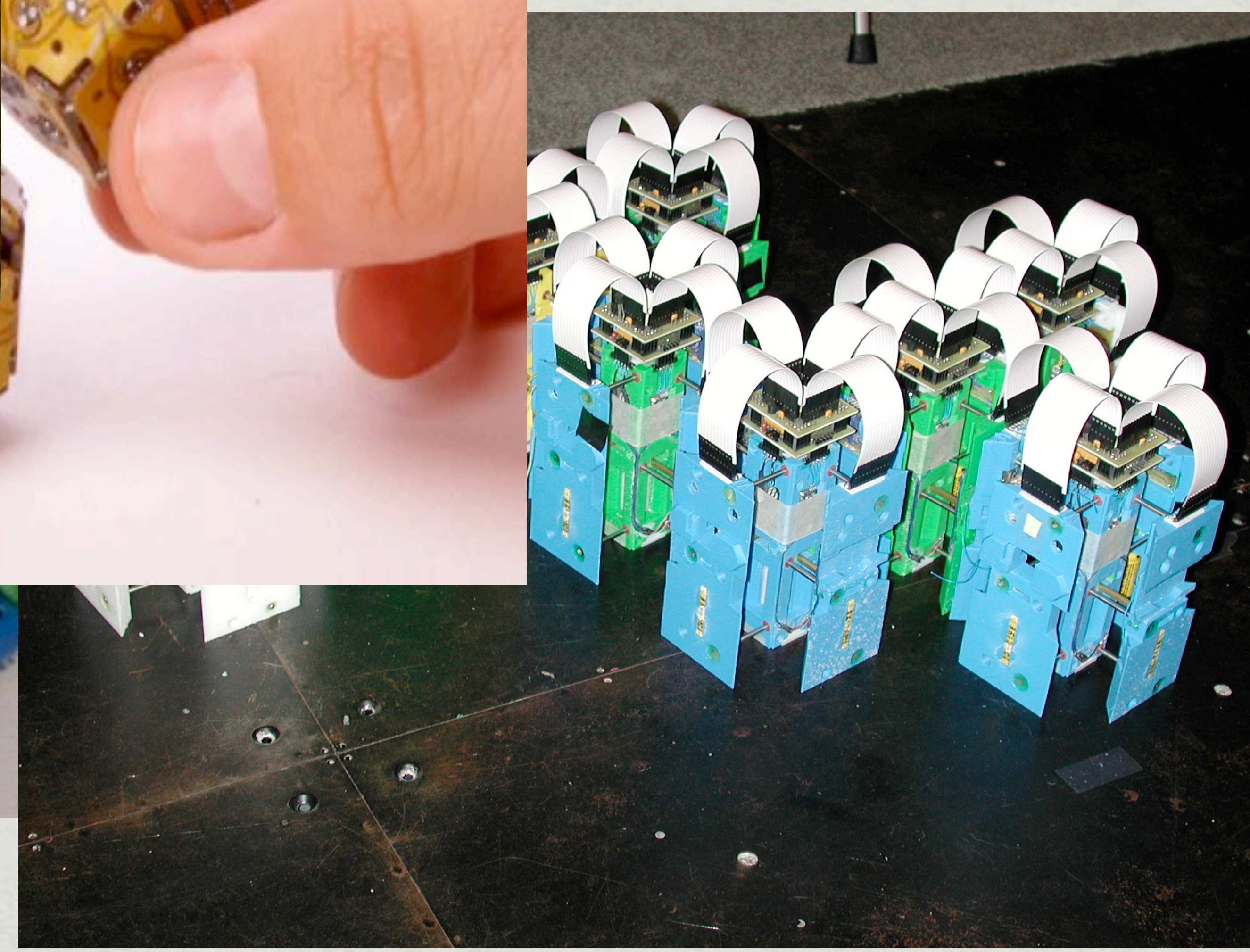
The space we build to capture these movements will also be appropriately discretized.

...from the world of robotics



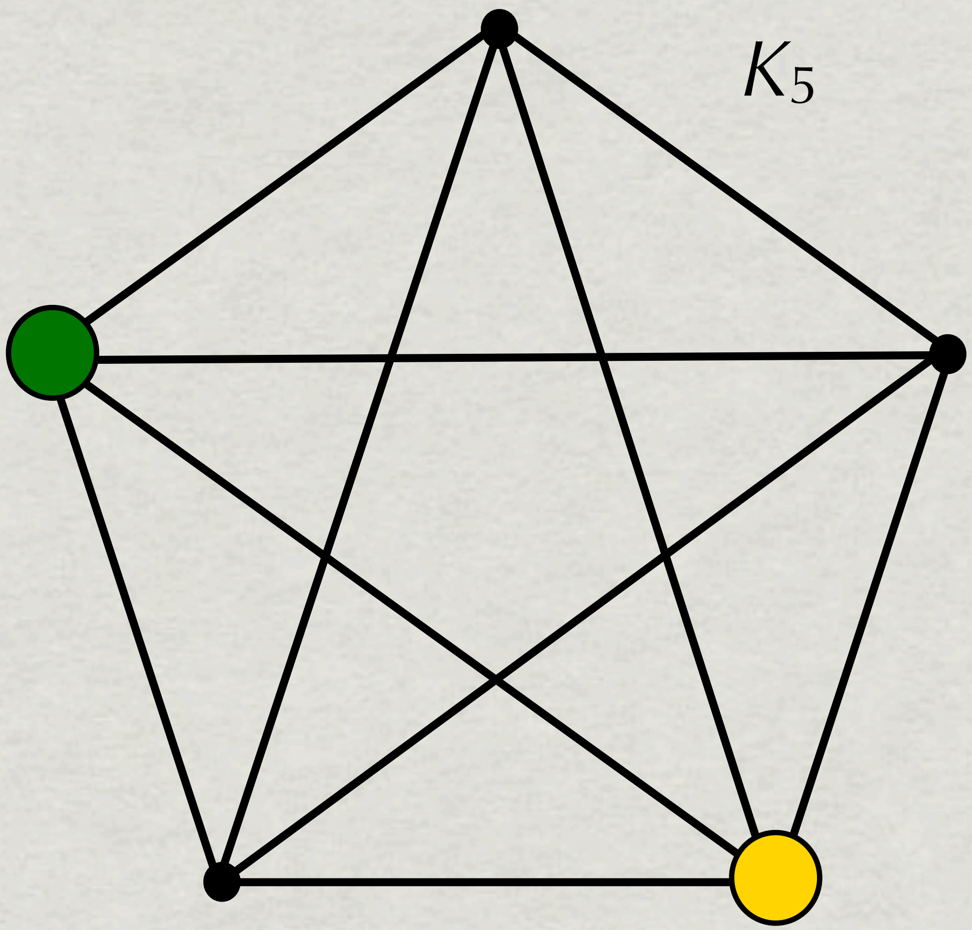
D. Rus, Distr. Rob. Lab, MIT
G. Chirikjian, JHU
S. Homans, M. Yim, PARC





Chapter 2: Definitions & Constructions

Two robots moving on a track



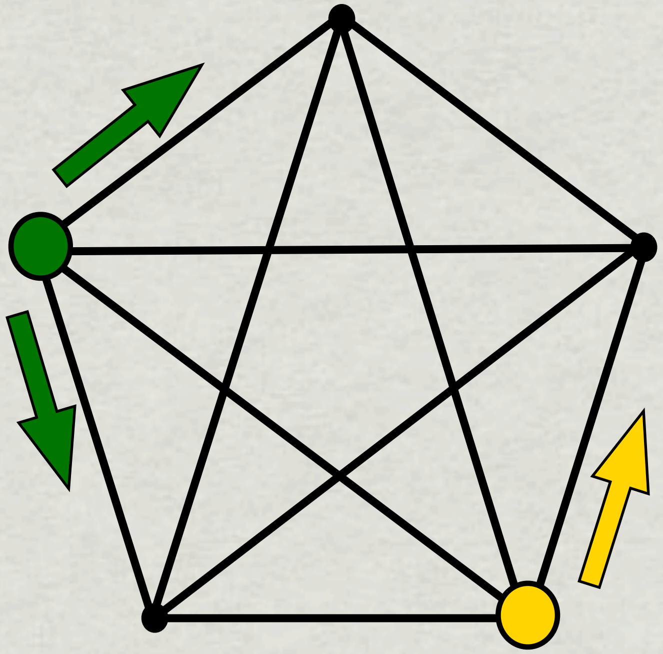
- Agents can slide along an edge to an empty vertex.
- No stopping, backing up, or communicating.

Definitions:

- A configuration of the robots on the vertices of the track is called a **state**.
- A move between states is called a **generator**.
- A closed collection of states and generators is called a **reconfigurable system**.

We want to build a space, *X*, that captures the states of our system and the moves (generators) between states.

The State Complex



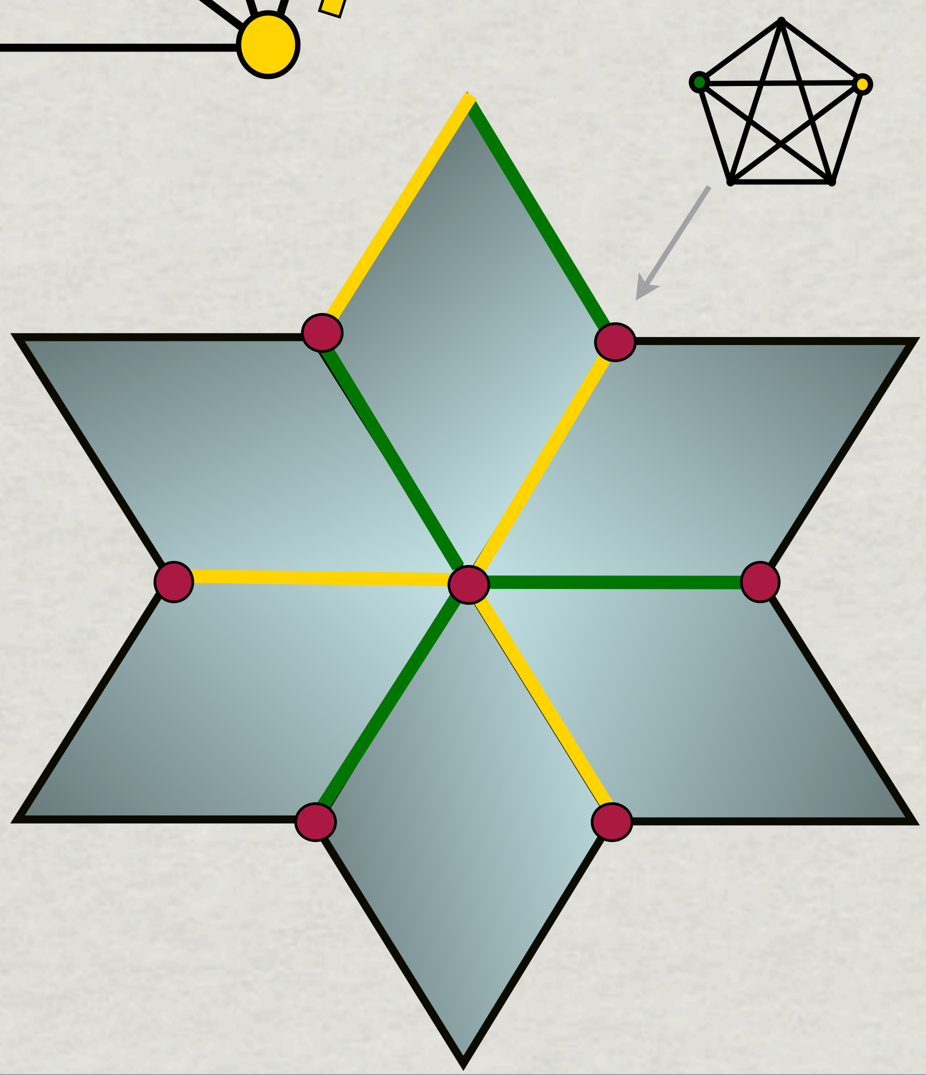
How do we build X?

transition graph = { **vertex** in X ⟷ **state**; **edge** in X ⟷ **generator**

We say these independent moves **commute,** and we capture this by adding information to X in the form of cubes:

square in X ⟷ **pair of commuting generators**

***k*-cube** in X ⟷ **collection of *k* pairwise commuting generators**

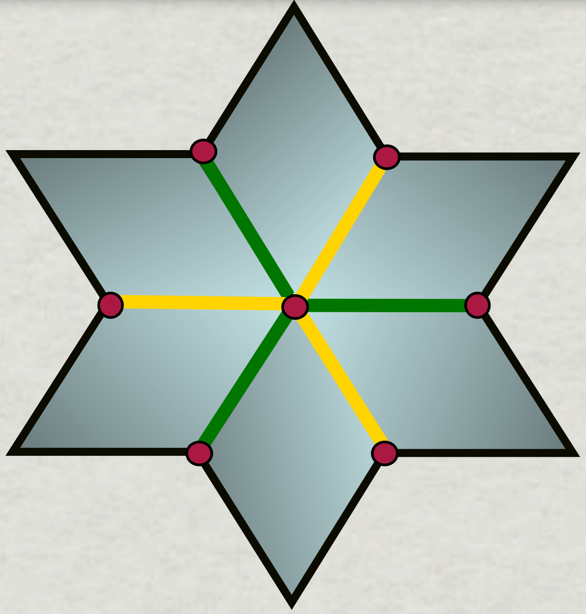


X

X is called the **state complex** for the reconfigurable system of two robots moving on K_5.

The State Complex

Let's finish our example:



- Due to symmetry in K_5 this local picture is repeated everywhere (i.e., every vertex in the state complex looks the same).

- Since squares patch cyclically around each vertex, gluing these local pictures together yields a closed (orientable) surface.

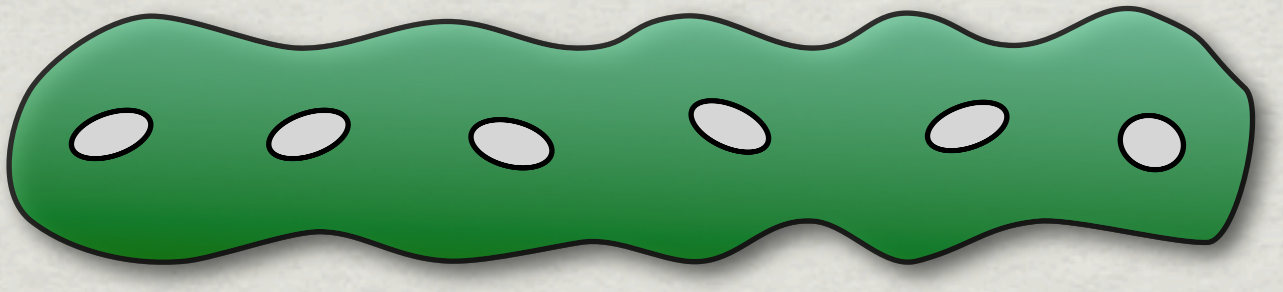
- Count: 20 vertices, 60 edges, and 30 faces in X.

  $\Rightarrow$ *Euler characteristic* $\chi(X) = 20 - 60 + 30 = -10$

- Since X is orientable, $\chi(X) = 2 - 2g$ (where g = genus, or # of handles).

  $\Rightarrow$ $-10 = 2 - 2g$

  $\Rightarrow$ $g = 6$ $\Rightarrow$ $X =$



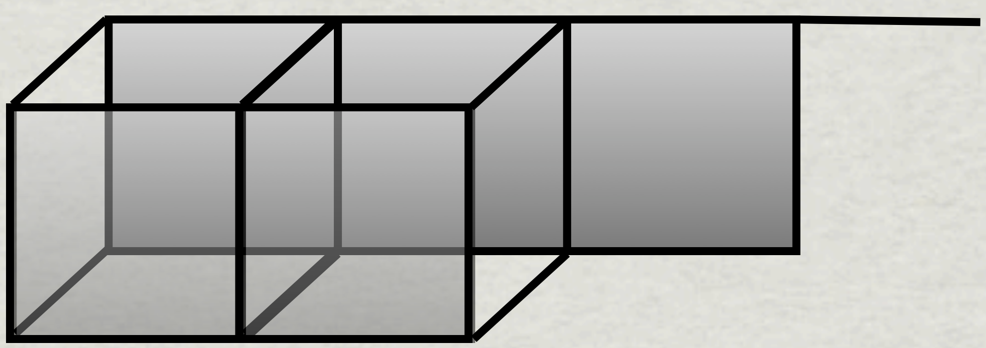
Cube Complexes

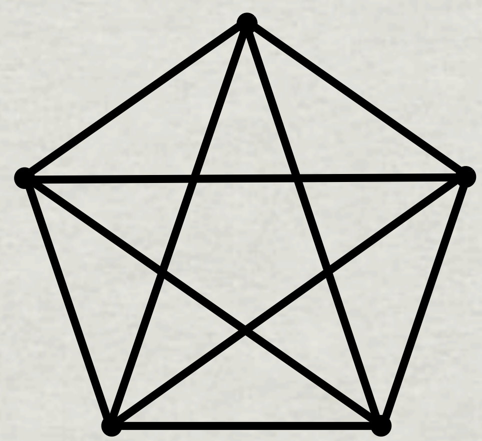
State complexes are examples of cube complexes.

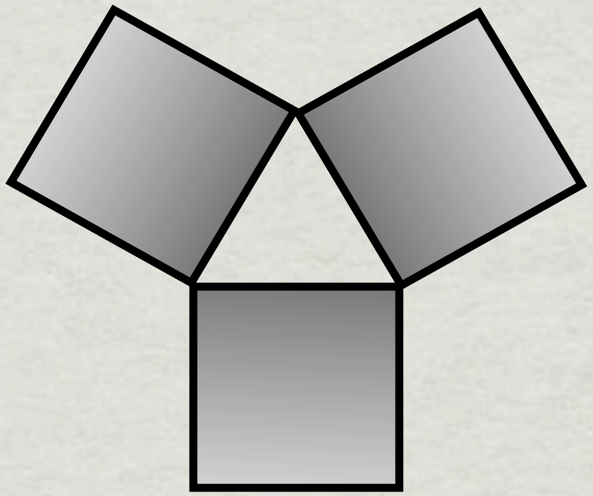
- A **cube complex** is a collection of cubes of the form $[-1, 1]^k$ glued together "nicely" along their boundaries (faces).

- Tools from smooth topology and geometry can be adapted so they apply to cube complexes.

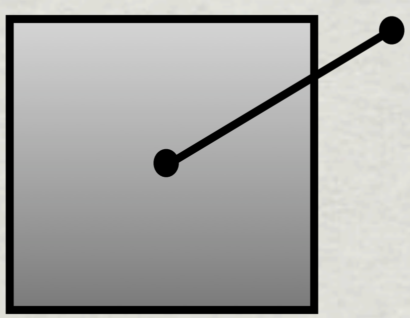
Examples:

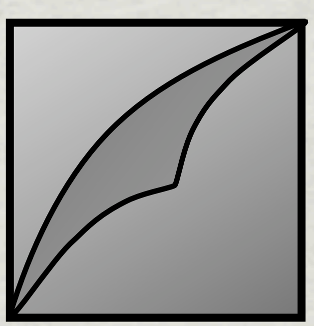






Non-examples:



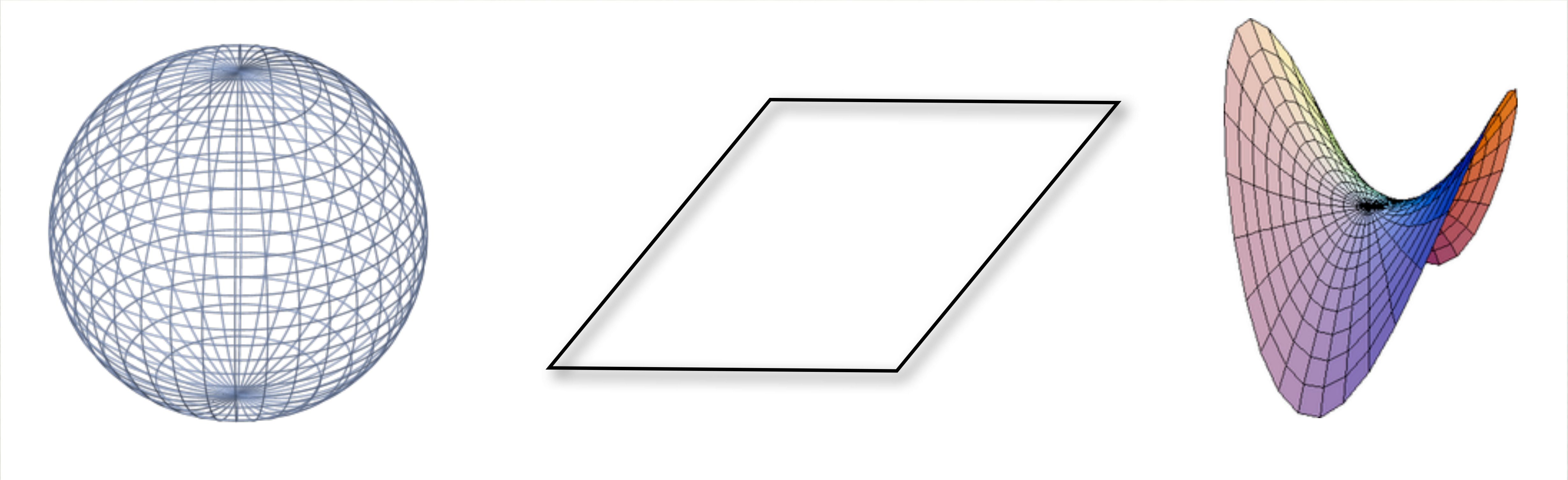


Chapter 3: Geometry, Topology, and Group Theory

Curvature 101

Geometers use the notion of curvature to measure how much a space deviates from being flat (or Euclidean).

✦ **Curvature in the model spaces**

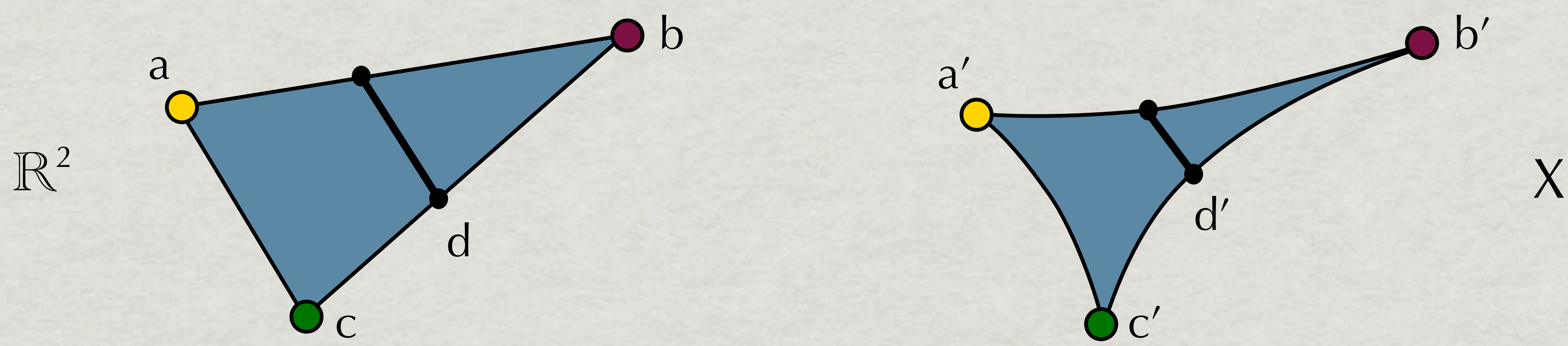


$\kappa > 0$ $\kappa = 0$ $\kappa < 0$

Curvature 101

Curvature in general metric spaces

- Rather than measure curvature directly, we often want to find an upper (or lower) bound on the curvature of a space.
- We can do so in any geodesic metric space, X, by comparing chords in geodesic triangles to chords in some comparison space.

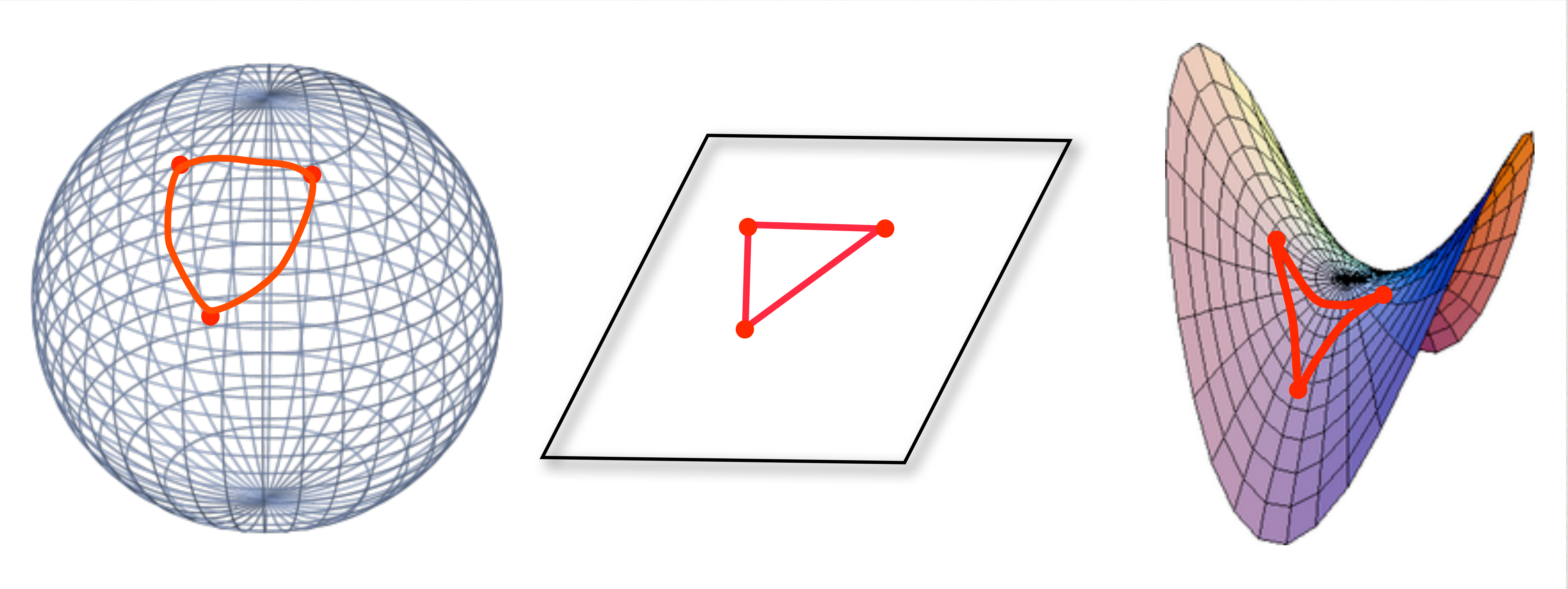


If d′ ≤ d for all chords in all triangles, we say X is **CAT(0)**.

If d′ ≤ d only in sufficiently small triangles, we say X is locally CAT(0), or **non-positively curved** (NPC).

Curvature 101

- Geodesic triangles in the model spaces



thin triangles
⇒ CAT(0)

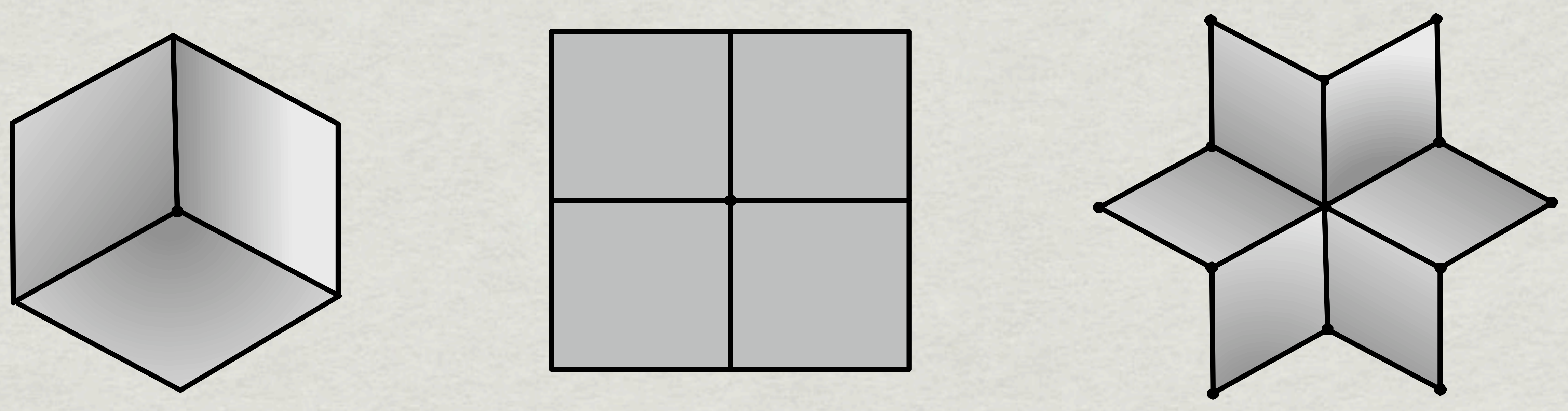
Curvature? What curvature?!

I thought we were talking about cube complexes. Aren't cubes FLAT?

They are... *in the interiors*. Curvature can be concentrated where several cubes come together. Here's how:

- **Curvature in the (cubulated) model spaces**

≤ 3 squares | 4 squares | ≥ 5 squares



$\kappa > 0$ | $\kappa = 0$ | $\kappa < 0$

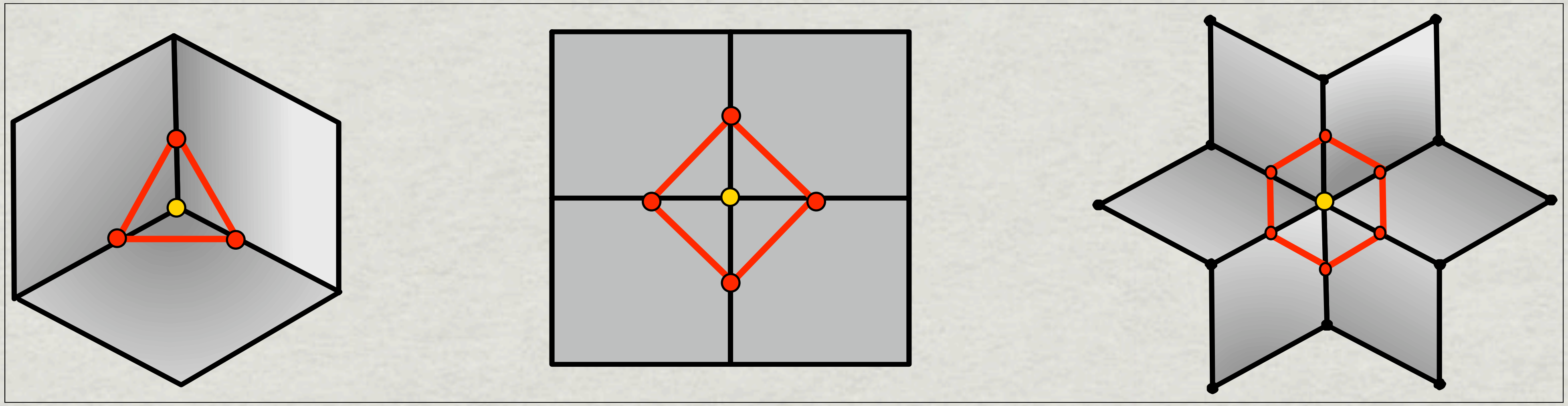
To find curvature bounds, all we have to do is check *all* chords in *all* geodesic triangles in our space and compare them to similar chords in a model space...

Curvature for cube complexes

A theorem of Gromov provides a combinatorial way to detect the presence of non-positive curvature in cube complexes.

Gromov's Link Condition: A cube complex is NPC ⇔ the **link** of each vertex is a **flag** complex.

the **link** of *v*, $lk(v)$:



A simplicial complex is **flag** if whenever edges bound a *k*-simplex, that *k*-simplex itself belongs to the complex (*i.e.*, all triangles are filled in)

Theorem [Ghrist, P]: State complexes are NPC.

Implications of NPC

- Spaces that are NPC have universal covers that are CAT(0) and therefore contractible. (In a CAT(0) space, geodesics are unique.)
- The higher homotopy groups of *X* vanish, so *X* is an Eilenberg-MacLane space, or a $K(\pi, 1)$ space.
- The fundamental group, $\pi_1(X)$, is torsion-free.

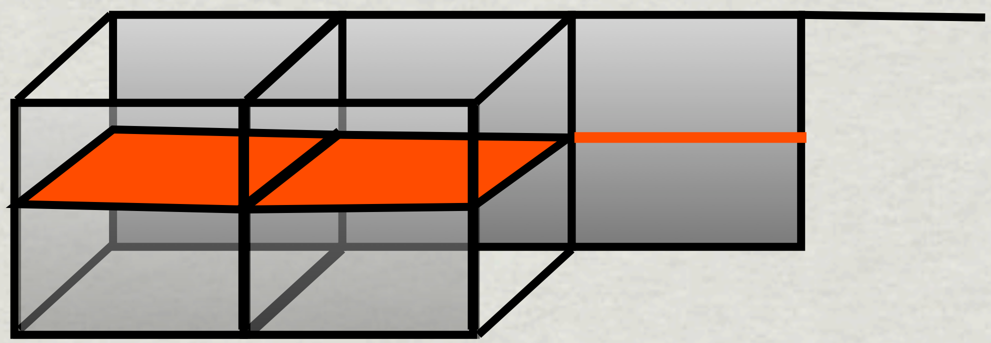
Moral: Geodesics exist in state complexes, and there's only one geodesic in each homotopy class.

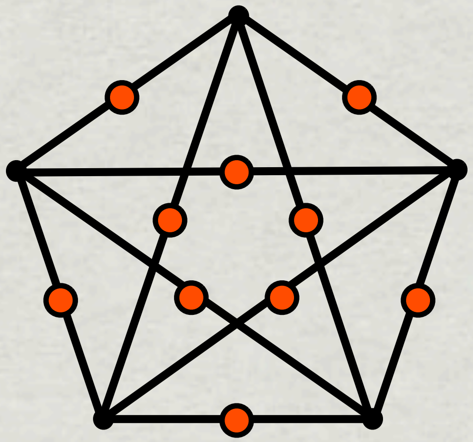
Thus, **finding optimal paths between configurations of our robots is not only *possible*, it's *not too hard***.

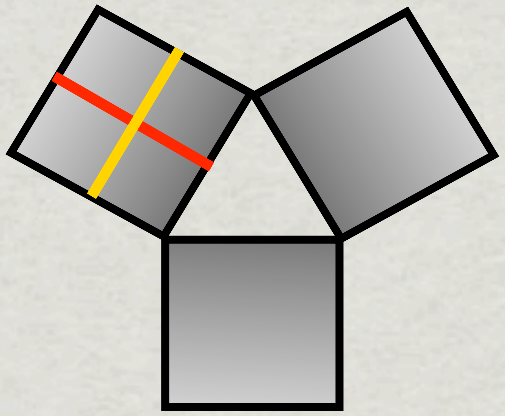
Group Theory & Topology (via more geometry)

✦ Hyperplanes

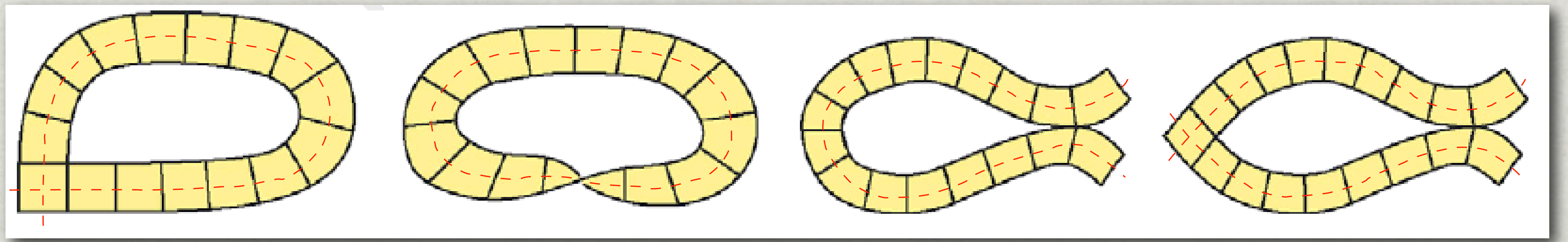
We can obtain information about our cube complex by looking at its **hyperplanes**: "slices" obtained by setting one coordinate $x_i = 0$.







✦ Badly behaved hyperplanes



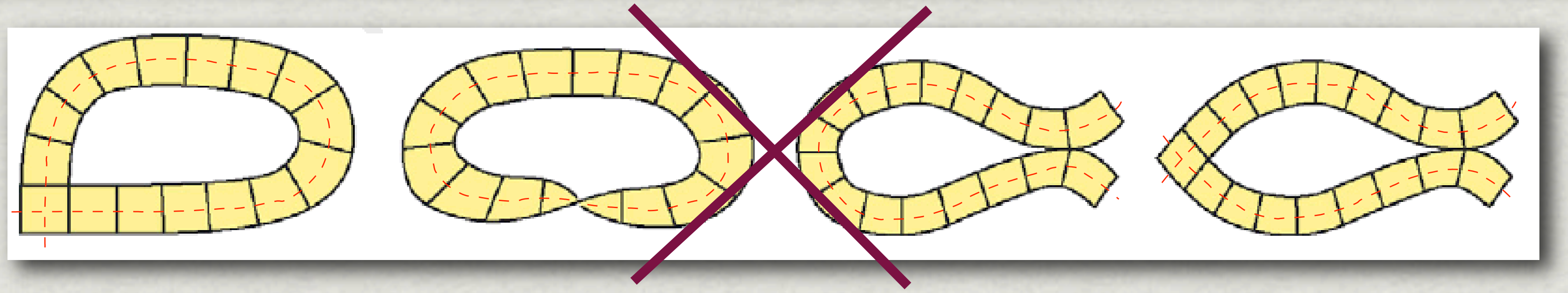
self-intersecting hyperplane

one-sided hyperplane

self-osculating hyperplane

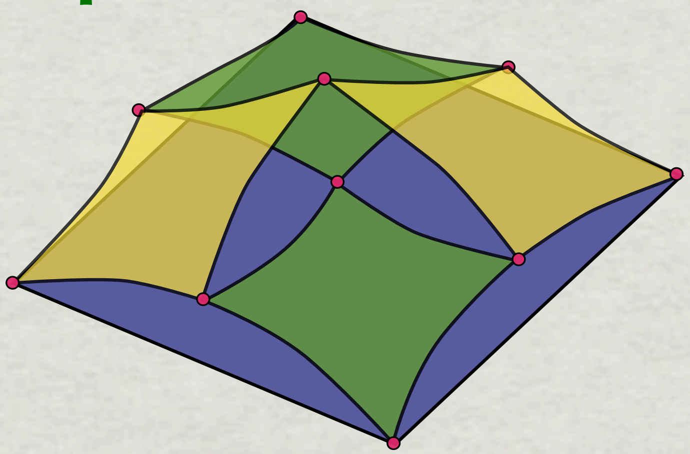
inter-osculating hyperplanes

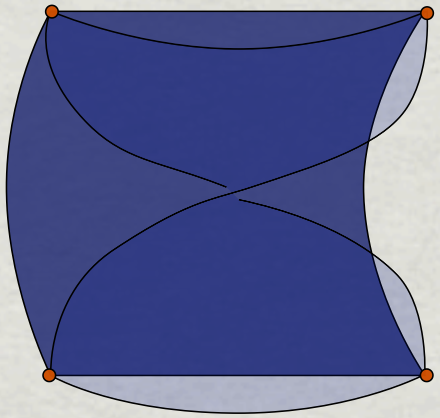
Group Theory (via more geometry)

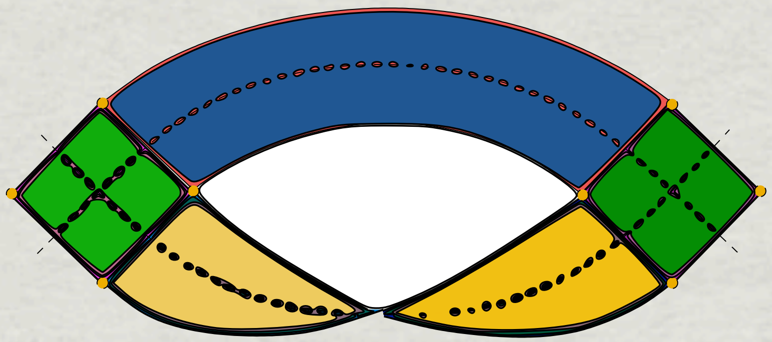


Definition: A cube complex that avoids these hyperplane pathologies is called **A-special**.

Examples:







Theorem [Ghrist, P]: **State complexes are A-special.**

Theorem [Ghrist, P]: **Fundamental groups of state complexes are subgroups of right-angled Artin groups.**

Implications of "special"

- **Right-angled Artin groups** are groups with the following presentation:

$$A = \langle a_1, a_2, \ldots, a_n \mid a_i a_j = a_j a_i \text{ for some set of } i \neq j \rangle$$

- Right-angled Artin groups are subgroups of linear groups, so fundamental groups of state complexes are in fact linear.

- Since these groups are finitely generated, they are **residually finite**.

  (Residually finite groups have lots of finite quotients, and so the spaces associated to them have lots of finite covers.)

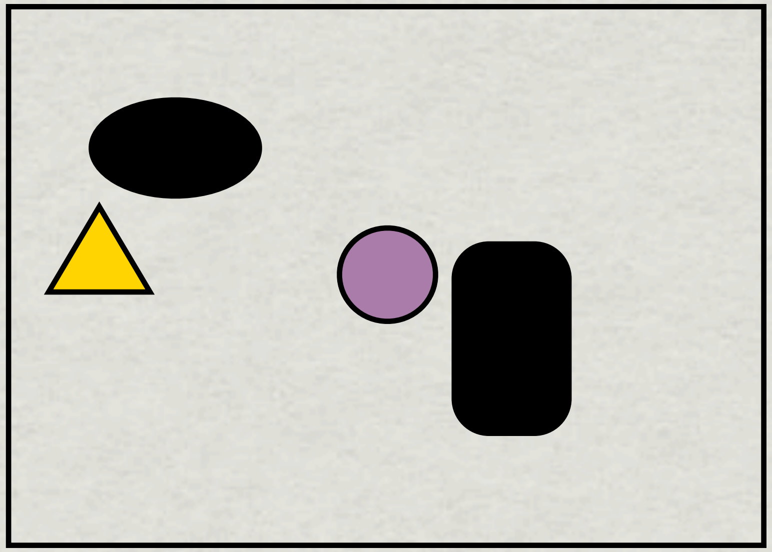
Moral: It is useful in geometric group theory to have examples of spaces that generate groups with these types of "finiteness properties."

From a topological standpoint, it can be useful to have many covering spaces that allow room to embed other spaces.

Chapter 4: Conclusions

Back to the beginning

Recall: We started our investigation in a factory.



While exploring the spaces that arise naturally in this context, we encountered rich and abstract mathematics from a variety of areas that were relevant to our investigation.

- There is a need for mathematical rigor in applications.
- There are a bevy of mathematical topics waiting to be applied.
- There is a lot left to be explored. → ***Get to work!***